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CONTENTS OF VOL. III,

From Fourier to Poincaré.

A Century of progress in Applied Mathematics.

(Inaugural address as Dr. Rashbehari Ghosh Professor of Applied Mathematics.*)

BY GANESH PRASAD

Introduction.

Sir Asutosh Mookerjee and Gentlemen,—I propose to address you this evening on the nature and extent of the progress in Applied Mathematics during the period which intervened between the award of the Grand Prize of the Academy of Sciences of Paris to Joseph Fourier in 1812 and the death of Henri Poincaré in 1912. The task that I have set before me is one of great difficulty, but I shall feel satisfied if I succeed in inoculating some of the young mathematicians, whom I see here, with germs of thought likely to develop into important and original researches.

For the sake of convenience, I will divide the address into two parts. In part I, I will mention some typical researches relating to the logical foundations of Applied Mathematics from the point of view of the believer in the continuous theory of matter as well as from that of the believer in molecules. Part II shall deal with the solutions of boundary-value problems.

PART I.

LOGICAL FOUNDATIONS OF APPLIED MATHEMATICS.

Continuous Theory.

1. I shall first consider what is commonly believed to be a simple and true theorem in the Dynamics of a particle. What is the criterion of the stability of equilibrium of a particle in a point (a, b, c), the force-function being $U(x, y, z)$? According to every well-known text-book on Dynamics, the answer is that U *must* have a maximum at (a, b, c). But this is wrong as has been pointed out by Professor Painlevé† who shows that, although for

$$U(x, y, z) = \tfrac{1}{2}\left(x^5 \sin\frac{1}{x} - y^2 - z^2\right),$$

$(0, 0, 0)$ is a position of stable equilibrium, $U(0, 0, 0)$ is not a maximum.

* This address was delivered at the Senate House on the 15th of December, 1914 before a meeting presided over by the Hon'ble Justice Sir Asutosh Mookerjee, C.S.I.

† *Comptes Rendus*, t. 138, 1904.

2. Let us now consider the linear flow of heat in an infinite slab, bounded by the impermeable faces $x = -\pi$ and $x = \pi$, and having its initial temperature equal to

$$\sum_{0}^{\infty} \frac{\cos (13^n x)}{2^n}.$$

Can we find the temperature at any point at time t? According to every text-book on the Conduction of heat, the answer is that the required temperature is

$$V = \sum_{0}^{\infty} \frac{e^{-13^{2n} t} \cos 13^n x}{2^n}.$$

But, to say the least, this solution is not unobjectionable, as was pointed out by me in a memoir published in 1903.* As a matter of fact, the initial value of $-\frac{\delta V}{\delta x}$, *i.e.*, the rate of flow of heat, becomes infinite at every point of an everywhere dense aggregate. For this reason, the initial temperature may be considered to be "inadmissible" and a solution impossible.

3. I proceed to bring to your notice a remarkable discovery of Prof. Hilbert. The famous law of Kirchhoff about the constancy of $\frac{\eta}{a}$, the ratio of emission to absorption, is no doubt "proved" in some sense in all well-known text-books on the theory of Radiation. But Prof. Hilbert† has recently pointed out that the law cannot be proved without the use of integral equations. Thus, according to the almost indisputable authority of Hilbert, the most important law in the elementary theory of Radiation remained without a proof until a few years ago when Prof. Hilbert proved it.

4. Before proceeding to the discussion of the molecular theory of matter, I should like to mention two recent discoveries relating to the nature of the potential-function. (*a*) About ten years ago, at the suggestion of Prof. Hilbert I undertook the careful consideration of Riemann's opinion, that every solution of

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} = 0$$

must be analytical, with the result that in 1906 I disproved* Riemann's statement by constructing a solution V, for which $\frac{\delta^2 V}{\delta x^2}$ and $\frac{\delta^2 V}{\delta y^2}$

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* *Abhandlungen d. k. Gesellschaft d. Wissenschaften zu Göttingen*, Bd. 2.

† *Jahresbericht d. d. Mathematiker-Vereinigung*, Bd. 22, *1913.*

are discontinuous at the points of an everywhere dense aggregate and which, therefore, cannot be expanded by Taylor's theorem in any one of these points. (*b*) The other discovery to be mentioned is due to Prof. Picard† who has shown that, contrary to the views of Gauss and Carl Neumann, in order that the potential-function, corresponding to a prescribed boundary value f (s), be expressible as the logarithmic potential due to a continuous distribution on the boundary, the continuity of f(s) and even the existence of f′(s) do not suffice.

Molecular Theory.

5. At the outset, I should like to sound a note of warning. Since in the sequel, I shall have to criticise somewhat severely such truly great men as Fourier and Maxwell, I must beg you to remember that, as Prof. Klein‡ said on a memorable occasion, progress in the logical foundations of Applied Mathematics can be made only by concentrated and independent thought and not by blindly following famous writers, whether they are of the type of Maxwell or Mach, Kirchhoff or Boltzmann, Fourier or Helmholtz.

In Fourier's "*Théorie analytique de la Chaleur*" and in Maxwell's "*Scientific Papers*" there are many places where the word "molecule" occurs. Shall we therefore conclude that these writers attempted to build up consistently any branch of Applied Mathematics on a molecular basis? The answer is emphatically, No. What they and a large number of their imitators did was, to start with the assertion that they would consider matter to be made up of molecules and then to ignore these molecules and treat them as points. As a matter of fact, the first successful attempt to build up a branch of Applied Mathematics on a molecular basis was made by me and the results are to be found in Part II of my Göttingen memoir. I start with definite suppositions relating to the size, shape and oscillations of the molecules in a solid, and then build up a theory of the conduction of heat which is professedly inexact, *the amount of error being capable of evaluation.* As shown by me, the error becomes less and less as the size of the molecule is taken to be less and less.

6. Another truly consistent investigation on a molecular basis has been recently published by Prof. Hilbert* who, by the use of integral

* *Mathematische Annalen*, Bd. 64, 1907.

† *Rendiconti d. circolo matematico d. Palermo*, t. 29, 1909.

‡ *Nachrichten d. K. Gesellschaft d. Wissenschaften zu Göttingen, Geschäftliche Mitteilungen*, 1901.

equations, obtains results, relating to the kinetic theory of gases, which are of the same character as my results. Prof. Hilbert shows that, starting with definite suppositions, we can deduce the various known theorems relating to gases by carrying the approximation sufficiently far.

PART II.

SOLUTIONS OF BOUNDARY-VALUE PROBLEMS.

7. This part of my address will not take as much time as the first part, for the simple reason that the results are not so startling in character as those which I have mentioned to you. In fact, the work of attacking new boundary-value problems and of solving them, after the style of Fourier, went on at a rapid pace after the publication of Fourier's "*Théorie analytique de la Chaleur.*" Thus we owe the determination of the stationary state of heat in an ellipsoid to Lamè and a large number of other important and similar results to many writers including Liouville and Mathieu, Green and Kelvin, Heine and Hobson. It is, however, worthy of notice that some problems, *e.g.*, the determination of the non-stationary state of heat in an ellipsoid, which baffled the efforts of Mathieu and Heine, are still unsolved.

8. An important class of boundary-value problems have been recently solved by Professors Sommerfeld† and Carslaw‡ by the use of Green's functions for the equation.

$$\nabla^2 V + k^2 V = 0.$$

As a very interesting discovery, I mention the connection established by Prof. Whittaker‖ between integral equations and the problem of the non-stationary state of heat in an elliptic cylinder.

CONCLUSION.

9. From what I have said during the course of this address, you will have no difficulty in deducing two facts. First, the progress in the logical foundations of Applied Mathematics has been very slow. As a matter of fact, there are only a few branches of Applied Mathematics whose logical bases have as yet been carefully and completely explored. The second fact is this: Up to a few years ago, our methods of solving boundary-value problems were, generally speaking, antiquated. Let us hope that new branches of Pure Mathematics, *e.g.* the theory of functions of real variables and the theory of integral equations, will, in the hands of coming generations of mathematical investigators, grow in importance as fit instruments for the consolidation and advancement of the knowledge of Applied Mathematics.

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* *Mathematische Annalen*, Bd. 72, 1912.

† *Jahresbericht d. d. Mathematiker—Vereinigung*, Bd. 21, 1913.

‡ See his papers in the *Proc, L. M. S*, Vols. 8 and 13; also *Math. Ann.* Bd. 75.

‖ *Proceedings of the International Congress of Mathematicians held at Cambridge. 1912.*

On the Modern Theory of Integration

BY

W. H. YOUNG.

1. In attempting to generalise the concept of integration so as to render it applicable to functions of a somewhat general discontinuous type, Riemann asked himself under what circumstances the analytical expression whose limit expressed the integral in the case of a continuous function continued to have an unique limit when the function is discontinuous. He stated that he did not quite completely prove the necessary and sufficient conditions that this should be the case. By fixing his attention on an analytical expression, instead of on the concept itself in question, Riemann, although the step he took was a considerable one, in a certain sense barred the way to future progress. It is true that he did not confine his attention to only bounded functions, but his treatment of the integration of even these functions was incomplete. Such a simple function as a bounded semi-continuous one has in general no integral in Riemann's sense.

It is remarkable that Lebesgue in his presentation of his epoch-making work on the Theory of Integration takes like Riemann an analytical expression and considers the limit of it. This expression differs essentially in character from that of Riemann and it involves in its expression the concept of content of a set of points. Now this content is nothing more nor less than the integral of a function of the same general type as the function considered, except that it is a two-valued function. In other words Lebesgue supposes first a theory of integration of such functions elaborated, and then uses this theory for that of the more general type of function.

In my own treatment of the subject I avoid in the first instance any analytical expression whatsoever, and employ considerations which appear to me to be the natural extension of those which led to the definition of the integral of a continuous function. All turns on the use of a method and of a principle. The method is that of regarding functions as generated from simple functions by means of monotone sequences, and the principle is that if a function is so traced out by a monotone sequence of functions whose integrals have been already defined, we may take the limit of these integrals to be the integral of the limiting function provided only this limit is the same whatever the monotone sequence may be.

We may if we please employ the principle only in order to define the integrals of semi-continuous functions and then show that we may approach any function, whose integration is to be considered, from below by upper semi-continuous functions, and from above by lower semi-continuous functions. It then appears that if the given function is to have an integral greater than that of all smaller and less than that of all larger functions, it is perfectly determinate being at once the upper bound of the integrals of upper semi-continous functions less than the function, and the lower bound of the integrals of lower semi-continuous functions greater than the function.

2. One of the great advantages of the treatment here indicated is that precisely the same ideas are involved equally whether the functions considered are functions of a single variable, or of several variables, and all the fundamental theories become intuitive.

For example if $f(x, y)$ is bounded, it is at once evident that

$$\int\int f(x, y)\, dx\, dy = \int\int f(x, y)\, dy\, dx$$

both being moreover equal to

$$\int f(x, y)\, (dx\, dy)$$

3. From this simple fact, we can show that Lebesgue's expression naturally presents itself as soon as we introduce the concept of content into the analytical treatment.

Indeed if $f(x)$ be positive

$$\int_a^b f(x)\, dx = \int_a^b \left[\int_0^{f(x)} dk \right] dx$$

Let k_0 be any quantity $>$ upper bound of $f(x)$ and let $\phi_k(x)$ denote the function of x which is unity when $k \leq f(x)$ and is zero elsewhere, then

$$\int_a^b f(x)\, dx = \int_a^b \left[\int_0^{k_0} \phi_k(x)\, dk \right] dx$$

$$= \int_0^{k_0} \left[\int_a^b \phi_k(x)\, dx \right] dk$$

$$= \int_0^{k_0} I(k)\, dk$$

where $I(k) = \int_a^b \phi_k(x)\, dx$ *i.e.* the integral of a function which $= 1$ where $f(x) \geq k$ and is zero elsewhere *i.e.* it is the content of the set of points at which $f(x) \geq k$. Now $I(k)$ is a montone decreasing function of k and $\therefore$ has only a countably infinite number of discontinuities; accordingly the Riemann expression is available for

$$\int_0^{k_*} I(k)\, dk.$$

But when this has been written down we recognise that it is equivalent to Lebesgue's.

4. Another advantage of the treatment adopted is that there is no sudden break in passing from the theory of integration of bounded to that of unbounded functions. We assume for simplicity of explanation that the functions considered are all positive. Then the fundamental theorem in this part of the theory will be the following :—

Theorem—If $f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots \longrightarrow f$ *be any monotone ascending succession of bounded functions having the unbounded function f for its limit, infinite values being allowed, then the limit of the integrals of the functions of the succession will be independent of the particular succession employed.*

We shall denote integrals of functions by capital letters.

Let $g_n = f$, wherever $f \leq n$ and let $g_n = n$, where $f \geq n$. We thus get a new monotone ascending sequence $g_1, g_2, \cdot \cdot g_n \cdots$ whose limit is f. It will suffice in order to prove the theorem to show that

$$\underset{n \longrightarrow \infty}{L_t} G_n = \underset{n \longrightarrow \infty}{L_t} F_n$$

Let $h_{p,n} = f_n$ wherever $f_n \leq p$ and $h_{p,n} = p$ where $f_n \geq p$;

then $h_{p,1}, h_{p,2} \cdots h_{p,n}$ form a monotone ascending sequence having g_p for limit, for $g_p = f$ where $f \leq p$, and is therefore the limit of f_n at such points, and accordingly is also the limit of $h_{p,n}$ at such points.

Elsewhere $h_{p,1}, h_{p,2} \ldots$ all have the value p, like g_p itself, from and after some place.

Hence since $h_{p,1}, h_{p,2} \ldots$ form a bounded sequence it follows by the earlier part of the theory that

$$\underset{n \longrightarrow \infty}{L_t} H_{p,n} = G_p$$

and therefore, since $f_n \geq h_{p,n}$ for all values of n, we have

$$\underset{n \to \infty}{Lt} F_n \geq G_p$$

But this is true for every value of .

Hence

$$\underset{n \to \infty}{Lt} F_n \geq \underset{p \to \infty}{Lt} G_p \ldots\ldots\ldots\ldots\ldots(1)$$

Again since f_n is bounded we can find an integer q such that $f_n \leq q$ everywhere.

Also $f_n \leq f$ everywhere.

Hence

f_n is everywhere less than the least of q and f

$$i.e.\ f_n \leq g_q$$

Hence

$$\underset{n \to \infty}{Lt} F_n \leq \underset{q \to \infty}{Lt} G_q \ldots\ldots\ldots\ldots(2)$$

Combining (1) and (2) we see that

$$\underset{n \to \infty}{Lt} F_n = \underset{n \to \infty}{Lt} G_n$$

Q.E.D.

5. It at once follows that if f is an unbounded function

$$\int_a^b f(x)\, dx = \underset{n \to \infty}{Lt} \int_a^b g_n\, dk$$

$$= \underset{n \to \infty}{Lt} \int_0^n I_n(k)\, dk$$

where $I_n(x)$ is the content of the set of points at which g_n is $\geq k$ $(k < n)$ and therefore is also the content of the set of points at which f is $\geq k$, for f and g_n only differ at points at which they are both $> k$ $(k < n)$.

Thus
$$\int_a^b f(x)\, dx = \underset{n \to \infty}{Lt} \int_0^n I(k)\, dk$$

$$= \int_0^\infty I(k)\, dk$$

where $I(k)$ is the content of the set of points at which $f(x) \geq k$.

6. The theory applies equally to functions which have both positive and negative values provided only that when the sign of the latter be changed the integrals of the function occurring exist in accordance with our definitions. Thus on this assumption the following general theorem is true:—

Theorem—*Term by term integration of monotone successions of functions is always allowable.*

By considering two auxiliary successions got by alternately putting all the positive values zero, and all the negative values zero, this theorem is seen at once to break up into two parts *viz* those of the statement that:—

Monotone ascending successions of positive functions and monotone descending successions of positive functions can always be integrated term by term.

The proof of the former of these parts resembles that of the fundamental theorem of §5 but is rather simpler. The truth of the latter may be deduced from that of the former, certain simple considerations with regard to points at which the functions are $+\infty$ being employed if required.

7. We here give the proof of the former part of the statement *viz*:—

$$\text{If } f_1 \leq f_2 \leq f_3 \leq \ldots\ldots\ldots \longrightarrow f$$

be a monotone ascending succession of positive unbounded functions having f as limit then $\underset{n \longrightarrow \infty}{Lt} F_n = F.$

Form as in § 5 the auxiliary monotone ascending succession

$$h_{p,1} \leq h_{p,2} \ldots\ldots h_{p,n} \ldots\ldots \longrightarrow h_p$$

where $h_{p,n}$ is the less of p and f_n everywhere, and h_p is the less of p and f

Then since this is bounded $\underset{n \longrightarrow \infty}{Lt} \underset{p,n}{H} = H_p$ Hence as before

$$\underset{n \longrightarrow \infty}{L_t} F_n \geq \underset{n \longrightarrow \infty}{L_t} H_{p,n} \geq H_p$$

and $\therefore \underset{n \longrightarrow \infty}{L_t} F_n \geq \underset{p \longrightarrow \infty}{L_t} H_p \geq F \ldots$ (1) by the fundamental theorem. But $F_n \leq F$ and $\therefore \underset{n \longrightarrow \infty}{L_t} F_n \leq F \ldots\ldots$ (2)

Combining (1) and (2) we have

$$\underset{n \longrightarrow \infty}{L_t} F_n = F$$

Q. E. D.

8. If $f_1 \geq f_2 \geq f_3 \ldots \geq f_n \ldots \longrightarrow f$ (1) be a monotone descending succession of positive functions having f for limit, then it can be changed into the ascending one

$f_1 - f_2 \leq f_1 - f_3 \leq \ldots \leq f_1 - f_n \ldots \longrightarrow f_1 - f \ldots\ldots (2)$ provided only none of the f's have the value $+\infty$. If they have such values we can attribute to all the functions $f_1, f_2 \ldots f_n, f$ the value zero wherever f_1 is $+\infty$ without affecting their integrals. The succession will then remain monotone, and the above treatment will be allowable. Moreover in either case by integrating the new succession (2)

we get $\underset{n \longrightarrow \infty}{L_t} (F_1 - F_n) = F_1 - F$

and therefore $\underset{n \longrightarrow \infty}{L_t} F_n = F$

Q. E. D.

9. Equally whether a function is bounded or unbounded, and everywhere of the same sign or not, we can, so long as it possesses an integral in the sense we have defined, approach it from below by upper semi-continuous and from above by lower semi-continuous functions each of which possesses an integral, and is such that the upper bound of the integrals of the former and the lower bound of the integrals of the latter are the same.

We thus obtain, using Scheefer's theorem that, if a derivate of a function is known to be greater than the same derivate of another function except at most a countable set of points then the former function exceeds the latter function by a monotone increasing function, an intuitive proof of one of the most general results in the modern theory of integration *viz.*

Theorem—*If a derivate of a function possess an integral and have the values $+\infty$ and $-\infty$ at most at a countable set of points then the integral of the derivate is the function.*

The proof indeed is immediate if we use the obvious facts that the derivates of the integral of an upper semi-continuous function are nowhere greater and those of a lower semi-continuous function are nowhere less than the semi-continuous function in question.

The function whose derivate is considered is accordingly greater than the integrals of all the upper and less than the integrals of all the lower semi-continuous functions considered and accordingly coincides with their common bound. In other words it is the integral of its derivate.

Aryyabhatta's rule in relation to Indetérminate Equations of the First Degree.

By

N. K. Mazumdar.

The principal object of this paper is to interpret, explain and compare Aryyabhatta's rule in relation to Indeterminate Equations of the First Degree of some such form as $Ax - By = C$, A and B being positive integers and C *any* integer. I shall also show that Aryyabhatta was not, so far as his rule indicates, in any way indebted to Euclid or other Greek or Alexandrian Mathematicians, as has been maintained by Mr. G. R. Kaye[1], and Heath[2], and others[3]. Incidentally I shall point out some other mistakes in the investigations of Mr. Kaye[1].

Aryyabhatta's works consist of 4 parts, the second of which is the *Ganita* or mathematical section proper, consisting of 33 couplets, the last two of which deal with the solution of Indeterminate Equations of the form I have indicated.

The rule[4] is :—" *The greater original divisor is divided by the lesser original divisor, and the rest divide one another. An assumed number together with the original difference is thrown in. The lower is multiplied by the upper and the last added.*

Divide by the smaller first divisor and multiply the remainder by the larger first divisor, add the original larger remainder for the final result "[5]

The technical terms of this rule may be best understood by considering the general problem, which most probably gave rise to it and which might be stated thus :—

To find a number n which will leave given remainders when divided by given positive integers; *i.e.*, n is to be found from the set of equations

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[1]. "Notes on Indian Mathematics. No. 2—Aryyabhatta" in the Journal of the Asiatic Society of Bengal, March, 1908.

[2]. Heath's *Diophantos of Alexandria.* New Edition, 1910 Heath bases his latest opinion on Mr. Kaye's paper, as he himself acknowledges in the footnote of P 281 of *Diophantos.*

[3] Chrystal, in his Algebra II.

[4] अधिकाग्रभागहारं छिन्द्यादूनाग्रभागहारेण ।
शेषपरस्परभक्तं मतिगुणमग्रान्तरे क्षिप्तं ॥ ३२ ॥
अधउपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेषं ।
अधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतं ॥ ३३ ॥

—The Âryabhatîya, Kern's ed.

[5] Quoted from Mr. Kaye's translation of Aryyabhatt's "Ganita"—32nd & 33rd couplets—published in the Journal of the Asiatic Society of Bengal, 1908.

$$\left.\begin{array}{l} n/A = x+R_1/A \\ n/B = y+R_2/B \end{array}\right\} \quad \mathrm{I}$$ [6]

which leads to the indeterminate equation

$$Ax-By = C \text{ (say).} \quad \mathrm{II}$$

[A numerical example corresponding to this may be taken from the problem of calendars :—

"The year 1 of Christian era was in the solar cycle (=28 years) the year 10, and in the metonic cycle (=19 years) it was 2. What was it in the Dionysian cycle (=28 × 19 years) ?

If n be the number in the Dionysian cycle—

$$n/28 = x+10/28$$
$$n/19 = y+2/19$$

whence $28x-19y = -8$"]. [7]

Here A, B are greater, or lesser original divisor, R_1, R_2 larger or lesser original remainder, and C is the original difference.

Only the first part of the rule will be discussed in this paper.

1. Let B < A. Then the modus operandi will be as follows—

$$\begin{array}{llllll}
\mathrm{B})\ \mathrm{A}\ (a_1 & & & & & \\
\underline{a_1\ \mathrm{B}} & & & & & \\
r_1)\ \ \mathrm{B}\ \ (a_2 & & & & & \\
\quad \underline{r_1\ a_2} & & & & & \\
\quad r_2)\ \ r_1\ \ (a_3 & & & & & \\
\qquad \underline{r_2\ a_3} & & & & & \\
\qquad r_3)\ \ r_2\ \ (a_4 & & & & & \\
\qquad\quad \underline{r_3\ a_4} & & & & & \\
\qquad\quad \vdots & & & & & \\
\qquad\quad \overline{\qquad\quad}(a_n & & & & & \\
\qquad\quad \overline{\quad r_n \quad} & & & & &
\end{array}$$

To simplify the matter suppose there are 4 quotients only, *i.e.*, the process terminates with a_4 ; so that $r_3 = 1$, $r_2 = a_4$. Then according to the first part of the rule we must set down the series

$$\begin{array}{lllll}
a_1 & & & & \\
& a_2 & & & \\
& & a_3 & & \\
& & & a_4 & \\
& & & & t \text{ (= an assumed integer)} \\
& & & & C.
\end{array}$$

[6] A century and a half later Brahmagupta also undertakes the solution of kindred problems

[7] Journal of the Asiatic Society of Bengal, 1908,—Mr. Kaye's Notes on Indian Mathematics.

from which may be derived the successive series

$$\begin{array}{llllll} a_1 \\ & a_2 \\ && a_3 &&& (1) \\ &&& t\,a_4 + C = D \text{ (say)} \\ &&& \quad t, \end{array}$$

$$\begin{array}{lllll} a_1 \\ & a_2 \\ && D a_3 + t = E \text{ (say)} && (2) \\ && \quad D, \end{array}$$

$$\begin{array}{llll} a_1 \\ & E a_2 + D = F \text{ (say)} && (3) \\ & \quad E, \end{array}$$

$$\begin{array}{lll} F a_1 + E = G \text{ (say)} && (4) \\ \quad F. \end{array}$$

Then, the values of G & F being actually calculated, the solution of II is—

$$y = G = At + C\,(a_1 + a_3 + a_1 a_2 a_3)$$
$$x = F = Bt + C\,(1 + a_2 a_3)$$

2. Now, evaluating the C. F. (continued fraction)

$$\frac{p_5}{q_5} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{C}{t}$$

by the ordinary arithmetical method, *i.e.*, from the end and not from the beginning, the successive steps are—

$$\frac{t\,a_4 + C}{t} = \frac{D}{t} \qquad (1)$$

$$\frac{t}{t\,a_4 + C} = \frac{t}{D} \qquad (1_a)$$

$$\frac{D\,a_3 + t}{D} = \frac{E}{D} \qquad (2)$$

$$\frac{D}{D\,a_3 + t} = \frac{D}{E} \qquad (2_a)$$

$$\frac{E\,a_2 + D}{E} = \frac{F}{E} \qquad (3)$$

$$\frac{E}{E\,a_2 + D} = \frac{E}{F} \qquad (3_a)$$

$$\frac{F\,a_1 + E}{F} = \frac{G}{F} \qquad (4).$$

therefore, $p_5 = G, \qquad q_5 = F.$

Now if the steps (1_a), (2_a), (3_a) be omitted, the remaining ones give the mode of operation corresponding to Aryyabhatta's rule. Thus Aryyabhatta practically finds

$$\frac{p_5}{q_5}=a_1+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{C}{t}$$

when $$\frac{p_4}{q_4}=\frac{A}{B}=a_1+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4},$$

and says $y = p_5$, $x = q_5$.

But what is the modern method?

The C. F. $\frac{p_3}{q_3}=a_1+\frac{1}{a_2}+\frac{1}{a_3}$

is evaluated,

and since $A\, q_3 - B\, p_3 = p_4 q_3 - p_3 q_4 = (-1)^4$

$$\therefore A\,(C\,q_3) - B\,(C\,p_3) = C$$

$$\therefore \text{also } A\,(Bt + Cq_3) - B\,(At + Cp_3) = C,$$

and therefore the general values of x & y are

$$x = Bt + Cq_3, \qquad y = At + Cp_3.$$

But $$\frac{p_5}{q_5} = \frac{t\,p_4 + C\,p_3}{t\,q_4 + C\,q_3} = \frac{At + C\,p_3}{Bt + C\,q_3}$$

$$\therefore p_5 = At + C\,p_3$$

$$q_5 = Bt + C\,q_3.$$

Thus Aryyabhatta's rule directly gives the most general values of x and y, and his conception of the assumed number — about 15 centuries ago — is simply wonderful.

3. Now what is the rationale of this rule? Of course the rule has come down to us in a written form, but as to explanations of the technical terms or any commentary on the rule itself by the author, there is none. Most probably these latter were orally transmitted from preceptor to pupil and the rule only was embodied in writing in a "Sutra" form—no doubt due to paucity of materials of writing in those early days. The following rationale[8] of the rule is advanced, which may be taken for what it is worth.

If $Ax-By=C$, then

$$\frac{Ax-C}{B}=y \text{ (an integer)} \qquad \text{(i)}$$

$$\therefore \frac{(a_1B+r_1)x-C}{B} = y\ (=G).$$

[8] Suggested to the writer by some commentaries by Chrishna of Bhaskaracharyya's 'Lilavati' and 'Vija-Ganita'.

$$\therefore \frac{r_1x-C}{B}=y-a_1x=y_1 \text{ (an integer)}$$

$$\therefore \frac{By_1+C}{r_1}=x \qquad \text{(ii)}$$

$$\text{or } \frac{(a_2r_1+r_2)y_1+C}{r_1}=x \ (=F).$$

$$\therefore \frac{r_1x_1-C}{r_2}=y_1 \qquad \text{(iii)}$$

$$\text{or } \frac{(r_2a_3+r_3)x_1-C}{r_2}=y_1(=E)$$

$$\text{or } \frac{r_2y_2+C}{r_3} = x_1 \ (=D) \qquad \text{(iv)}$$

But ultimately r_n must be $= 1$ (for by implication A and B are prime[9] to one another), and in the case under supposition

$r_3 = 1$, $r_2 = a_4$; therefore (iv) becomes $a_4y_2+C = x_1$ (iva)

Now y_2 assuming any arbitrary value t,

$$x_1 = a_4t + C,$$

and then by retracing the steps y_1, x, y may be obtained.

Thus the solution of (i) is made to depend upon the solution of a number of other inderterminate equations, the last of which, having one of its divisors equal to 1, is the simplest and can be readily solved. In fact x_1, y_1, x, y are actually the same as D, E, F and G respectively.

4. Mr. Kaye in his notes[10] on these two couplets says that, "The fundamental process involved in the method given by Aryyabhatta "is contained in the first and second propositions of the seventh book, and "the second and third of the tenth book of Euclid. The results of these "propositions translated into Algebraic notation give us the following "indeterminate equations: $AL-BM=1$ and $AL_1-BM_1=g$. The "process by which the former of these is arrived at may be exhibited " thus:—

$$\begin{array}{llll}
B) & A & (a_1 & \\
 & \underline{a_1B} & & \\
 & r_1) & B & (a_2 \\
 & & \underline{r_1a_2} & \\
 & & r_2) & r_1 \ (a_3 \\
 & & & \vdots \\
 & & & \overline{}\ (a_{n+1} \\
 & & & \underline{r_n}
\end{array}
\qquad \text{or } \frac{A}{B} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dotsb$$

[9] Among the ancients we first come across the statement that "If A and B are prime to each other, their H. C. F. must also divide C; otherwise the question is an impossible one" in the 1st rule of Bhaskaracharyya 5 centuries earlier than Bache gave his solution with the same condition in 1624 A.D.

[10] Journal of the Asiatic Society of Bengal, March, 1908.

"represents the process of finding the G. C. M. of two numbers A and "B. If the last remainder is unity, Euclid states that the two numbers "A and B are prime *interse.* His proof may be set down as follows :—

$$r_1 = A - a_1 B$$
$$r_2 = A(-a_2) + B(1 - a_1 a_2)$$
$$r_3 = A(1 + a_2 a_3) + B(-a_1 - a_3 - a_1 a_2 - a_3)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$r_n = (-1)^{n+1}(AL - BM)$$

"If $r_n = 1$ then A and B are prime to each other: for if not etc. If "$Ax - By = C$ and $AL - BM = 1$, we get $ALC - BMC = C$ and "$A(x - LC) - B(y - MC) = 0$ or $(x - LC)/(y - MC) = B/A$; and further "as A and B are prime to each other, we have $x - LC = Bt$ and $y - MC = At$ (Euclid VII, 33), where t is any integer.

"In solving $Ax - By = C$, where $r_3 = 1$, we have

"$A = a_2 + a_4 + a_2 a_3 a_4$

"$B = 1 + a_1 a_2 + a_1 a_4 + a_3 a_4 + a_1 a_2 a_3 a_4$

"$x = Bt + LC = t(1 + a_1 a_2 + a_1 a_4 + a_3 a_4 + a_1 a_2 a_3 a_4)$
$\qquad - C(a_1 + a_3 + a_1 a_2 a_3)$

"$y = At + MC = t(a_2 + a_4 + a_2 a_3 a_4) + C(1 + a_2 a_3)$

"Now, following Aryyabhatta's instructions, set down

$$\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ ta_4 - C \\ t \end{array}$$

"Add the lowest term to the product of the two proceding $t +$ "$a_3(t\, a_4 - C)$; multiply this result by the next highest term (a_2) "and add to the product the penultimate term $(t\, a_4 - C)$ and so on. "The final result in this case is $t(1 + a_1 a_2 + a_1 a_4 + a_3 a_4 + a_1 a_2 a_3 a_4)$ "$- C(a_1 + a_3 + a_1 a_2 a_3)$ which equals $Bt + LC = x$ as above.

"As t is any integer we may substitute any other integer for it. "Set $t_1 — = ta_r - C$ then $t = (t_1 — + C)/a_r$ and we have the series a_1, a_2, "$a_3, \ldots\ldots\ldots a_{r-1}$, $t_1 —$, $(t_1 — + C)/a_r$, which may be treated as before."

As regards this note, with all respect to Mr. Kaye I cannot but differ from him on the following points :—

(i) r_n is not the remainder after the quotient a_{n+1} but after a_n ; and

$$\frac{A}{B} \neq, \text{ but } \frac{B}{A} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots\cdots}}}$$

But surely these are minor points and most probably printing mistakes.

(ii) He also takes the case of the equation $Ax-By=C$ as I have done; and $r_3=1$ under his supposition also. But then the values of A and B as in this note should be interchanged, as also that of L and M. I fail to understand also why the sign of C has been abruptly changed in the value of x. The solution is practically that of $Bx-Ay=-C$.

(iii) Assuming that there is mistake, for as his note implies there should be a 't' just below 'ta_1-C,' I do not see how $\underline{ta_1-C}$ has been obtained and why the sign of C has been consistently changed throughout.

It should be ta_1+C in the case under supposition. Moreover the series, as it stands in the note, is not the original series, but ought to be the first deduced series, so far, of course, as Aryyabhatta's rule indicates.

(iv) Then there is the unnecessary trouble of substituting a new integer for t. I do not think Arryyabhatta's rule implies anything of the sort.

(v) This is the most important point of difference. It is maintained in the Notes that not only is there an analogy in the working of this rule and an "easy development" of Euclid's analysis of the G. C. M., but that it is practically contained in that "easy development." Heath also says,[11] "Thus the solution of the equation $ax-by=c$ given by Arryyabhatta (born 476 A.D.) as well as by Brahmegupta and Bhaskara, though it anticipated Bachet's solution which is really equivalent to our method of solution by C. F. is an easy development from Euclid's method of finding the G. C. M. or proving by that process that two numbers have no common factor (Euclid VII, 1, 2., X 2, 3), and it would be strange if the Greeks had not taken this step."[12]

Marshall [13] has well said that "facts are suggestive in their similarities, but are still more suggestive in the differences that peer out through those similiarities."

In this case the difference shall be more striking than the similarity. In the 'easy development' the conception of the assumed number comes after the ordinary values of the variables have been found; but as to this rule it is quite otherwise, for here that conception comes at the very beginning, without which the first step is impossible,

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[11] Heath, *Diophantos*, 2nd Edition, 1910, P. 281.

[12] This statement is based upon the investigations of Mr. Kaye, as Heath himself acknowledges in the footnote of the page.

[13] *Principles of Economics*, P. 379, f.

and which implies some other sort of analysis, as has been indicated. In the one case the ordinary values lead to the general ones, the ordinary values being themselves suggested by the method of Euclid; whereas in the other the general values are evolved first and the ordinary ones must be deduced from them. In fact the only similarity that can be traced between the two methods lies in the process of reciprocal division between A and B. But a striking difference peers out even through this similarity. For what is this similarity due to? In the case of the 'easy development' the result justifies the means but in the other the cause clearly explains the modus operandi. Thus this similarity has been the result of different forms of analysis, and is not the cause of the same sort of analytical reasoning.

The conception of the assumed number is thus more important than at first appears, for it clearly implies the existence of some sort of analysis in its truest sense behind the rule; and the rule does not necessarily depend upon Euclid's method, much less does it follow from it.

If the modern method be considered as simply based[14] on the theory of C.F., then of course it depends upon some result of Algebra fortunately obtained; and if that result were untrue, *i.e.*, if $p_n q_{n-1} - p_{n-1} q_n \neq (-1)^n$, but $=$ some other integer, say, k, then that analysis would fall to the ground and would have to be recast from the very beginning; but the same cannot be said of the sort of analysis as has been set forth in the preceding pages, for it is intrinsically correct so far as it goes.

Besides, the rule as given by Aryyabhatta stands unique up to the present time, for nowhere does Modern Algebra give the rule in this form, indicating to evaluate the $(n+1)$th convergent of the modified C.F. with n quotients, *directly* to yield the general values.

I do not know how far I have been able to lay sufficient stress on, or to draw sufficient attention to, the great importance of these two points.

5. Again Chrystal says[15] "Indeterminate equations the solutions of which are limited by extraneous conditions, are called Diophantine Problems, in honor of the Alexandrian Mathematician, Diophantos, who, so far as we know, was the first to systematically discuss such problems, and who showed extraordinary skill in solving them (See Heath's Diophantos of Alexandria, Camb. 1885—P. 473, f)."

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[14] See quotation from Heath's *Diophantos* above.

[15] Algebra Part II 2nd Edition, 1906, P. 473.

There is no denying that Diophantos was a great Mathematician but Chrystal was wrong if he intended to connect him with solutions of indeterminate equations of the 1st degree. The following may also be quoted from the same authority (*i.e.* Heath's *Diophantos*), only that it is a revised edition of 1910.

" Diophantos flourished about 250 A.D."[16]

"There is nothing to show that problems involving indeterminate equations of the first degree, formed part of the writer's plan."[17]

"Diophantos does not, in his *Arithmetica* as we have it, treat of indeterminate equations of the 1st degree"[18] in the sense in which they are ordinarily understood. Heath has taken the trouble to classify the problems of Diophantos, and there are in the whole of the *Arithmetica* only 3 cases of indeterminate equations of the 1st degree prpero, *viz.*,

(1)	$xy + (x + y) = a$	Lemma to IV, 34.
(2)	$xy - (x + y) = a$	„ „ IV, 35.
(3)	$xy = m(x + y)$	„ „ IV, 36.

But even in these examples Diophantos gives ' solutions in which y has been practically found in terms of x.'[19]

Colebrooke also says[20], " The general character of the Diophantine problems and the Hindu unlimited ones is by no means alike" etc.

Thus there is no foundation for the categorial statement of Chrystal.

In conclusion I have only to point out that " the general solution of indeterminate equations of the first degree was first given among moderns by Bachet de Meziriac in 1624 A.D." [21] Bhaskaracharyya had already given his elaborate rules for the general solution in the 1st half of the 12th century, and Brahmegupta contributed towards the same question about the middle of the 6th. But the work of Aryyabhatta (born 476 A.D.**) is the earliest in which we find the rule in a written form.

[16] Heath, *Diophantos*, 2nd Edition, 1910, P. 2.

[17] P. 7, Ib.

[18] P. 67, Ib.

[19] See Heath's *Diophantos*, PP. 192—4, 262.

[20] Colebrook's Translation of Lilavati, Introduction, PP. XV, XVI.

[21] Colebrooke, Lilavati, P. XVII.

** Notes on Indian Mathematics by Mr. Kaye in the Journal of Asiatic Society of Bengal, 1908. Also Heath, *Diophantos*, P. 281.

On the Figures of Equilibrium of a Rotating Mass of liquid for laws of attraction other than the law of Inverse Square, Part I.

By Bibhutibhushan Datta.

The present paper contains the first instalment of the results of my investigation of the figures of equilibrium of a rotating mass of a homogeneous incompressible liquid whose particles attract one another according to laws of force other than the Newtonian law of gravitation. In Art 1, I enunciate and prove the theorem, that if we consider the law of force to be given by $F = -\frac{\mu}{r^k}$, then it is only when $k = -1$ or 2 that an ellipsoidal figure can be a possible surface of equilibrium. In Art 2, I consider the figures for $F = -\frac{\kappa^3}{r^2} - \mu r$. In Art 3, I prove a theorem for the law of direct distance which is analogous to a theorem of Poincaré's for the law of inverse square of the distance.

The second part of my paper will contain a detailed discussion of Neumann's law and a discussion of the question of stability.

All the results in this paper are believed to be new.

I wish to express my indebtedness to Dr. Ganesh Prasad at whose suggestion I took up, and under whom I carried on, the investigation the results of which are embodied in this paper.

1. Let the law of force be the inverse kth power of the distance so that

$$F = -\frac{\mu}{r^k}.$$

Then the following new theorem holds true :—

It is only when $k=2$ or $k=-1$ that an ellipsoidal figure can be a possible surface of equilibrium of a mass of homogeneous incompressible fluid rotating about a fixed axis.

PROOF :—

If the ellipsoid rotate in relative equilibrium about the axis of z with an angular velocity ω, the dynamical equations are

$$-\omega^2 x = -\frac{1}{\rho}\frac{\delta p}{\delta x} - \frac{\delta \Omega}{\delta x}$$

$$-\omega^2 y = -\frac{1}{\rho}\frac{\delta p}{\delta y} - \frac{\delta \Omega}{\delta y}$$

$$o = -\frac{1}{\rho}\frac{\delta p}{\delta z} - \frac{\delta \Omega}{\delta z}$$

where $-\Omega$ is the potential.

The surfaces of equal pressure are therefore given by

$$\Omega - \tfrac{1}{2}\,\omega^2\,(x^2+y^2) = \text{const.} \qquad (1)$$

(a) Let the force vary directly as the distance, so that $k = -1$. Evidently then, the potential of the ellipsoid at a point (x, y, z) is

$$-\Omega = -\tfrac{1}{2}\,\mathrm{M}\mu\,(x^2+y^2+z^2) + \text{a constant.} \qquad (2)$$

where $\qquad \mathrm{M} = \tfrac{4}{3}\,\pi\rho\, abc.$

Substituting in (1), we have

$$x^2\,(\tfrac{2}{3}\,\mu\,\pi\,\rho\, abc - \tfrac{1}{2}\,\omega^2) + y^2\,(\tfrac{2}{3}\,\mu\,\pi\,\rho\, abc - \tfrac{1}{2}\,\omega^2)$$
$$+\ \tfrac{2}{3}\,\mu\,\pi\,\rho\, abc\, z^2 = \text{const.} \qquad (3)$$

In order that this may coincide with the external surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

we must have

$$a^2\,(\tfrac{2}{3}\,\mu\,\pi\,\rho\, abc - \tfrac{1}{2}\omega^2) = b^2(\tfrac{2}{3}\,\mu\,\pi\,\rho\, abc - \tfrac{1}{2}\omega^2) = c^2 \times \tfrac{2}{3}\,\mu\,\pi\,\rho\, abc \qquad (4)$$

Hence $\qquad a = b \qquad (5)$

and $\qquad \tfrac{2}{3}\,\mu\,\pi\,\rho\, abc\,(a^2-c^2) = \tfrac{1}{2}\,a^2\,\omega^2. \qquad (6)$

This will give a possible value of ω if $\quad a > c.$

Thus,* a planetary ellipsoid is a possible figure of equilibrium; the ovary ellipsoid cannot be a figure of equilibrium. Also, the axis of rotation is the smallest axis.

(b) When the force varies as the inverse square of the distance, the case has been studied by Maclaurin and Jacobi and the results are well known

(c) Now let us consider the general case when the force varies as the inverse kth power of the distance, where k is any integer, even or odd, positive or negative.

* See Besant and Ramsey's *Hydromechanics*, Art. 187

Dr. Routh has found the following formulæ for the potential V of a solid homogeneous ellipsoid at an internal point for such a law of force*:—

(i) When k is an odd negative integer

$$V = -\frac{4\pi abc}{2t}\left\{\frac{1}{3}+\frac{\nabla}{5\cdot 3!}+\frac{\nabla^2}{7\cdot 5!}+\ \ldots\ldots\right\}(x^2+y^2+z^2)^t \quad (7)$$

where $\nabla = a^2\frac{d^2}{dx^2}+b^2\frac{d^2}{dy^2}+c^2\frac{d^2}{dz^2}$

and $2t = 1-k$.

(ii) When k is even and > 4

$$V = -\frac{2\pi}{(k-1)\,(k-3)!}\left(\frac{2}{E}\right)^{k-4}\Sigma\frac{(k-5-f)!}{(f)!}\cdot\frac{E^f\,\nabla^f}{2^s}.\,P \quad (8)$$

where Σ implies summation from $f = 0$ to $\frac{1}{2}(k-4)$,

$$E = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2},$$

$$P = \left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{1}{2}(k-4)}.$$

(iii) When k is even, < 4 and may be negative

$$V = \frac{\pi\,(-1)^t}{(k-1)(t+1)!}\int_0^\infty v^{-\frac{1}{2}}\,dv\left(\frac{d}{dv}\right)^t\frac{R^{t+1}}{Q} \quad (9)$$

where $R = 1-\frac{v\,a\,x^2}{a+v}-\frac{v\,\beta\,y^2}{\beta+v}-\frac{v\,\gamma\,z^2}{\gamma+v}$,

$Q^2 = (a+v)\,(\beta+v)\,(\gamma+v)$,

$t = \frac{1}{2}(2-k),\ a,\beta,\gamma = \frac{1}{a^2},\ \frac{1}{b^2},\ \frac{1}{c^2}.$

From these formulæ we find that for any even value of k from 6 upwards,† and for any value of k from—2 downwards, the expression for V is other than a quadratic. Hence a quadratic surface cannot be a possible form of equilibrium of a rotating mass of homogeneous incompressible fluid for these laws of force.

* Phil. Trans. Roy. Soc., Vol. 186 A, (1895)

† Values of k from 3 upwards may be excluded also because for such values the potential becomes infinite on the surface.

(*d*) A very interesting case is when $k=o$, so that the force is independent of the distance. The potential of a solid ellipsoid at an internal point in this case is

$$V = \tfrac{1}{5} \iint (r_1^{4} + r_2^{4})\, dw \tag{10}$$

where r_1 and r_2 are the roots of the equation.

$$r^2 \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right) + 2r\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right) - \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right) = o \tag{11}$$

This will give a quartic expression for V.

(*e*) Let $k = 1$, *i.e.*, let the force vary inversely as the distance. Then, for a point on the surface,

$$-\frac{\delta\Omega}{\delta x} = 2\iint \frac{l\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2}{\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)^2}\, dw, \tag{12}$$

the integration being carried over a *hemisphere*.

If the ellipsoidal form is to be a figure of equilibrium $-\frac{\delta\Omega}{\delta x}$ must be proportional to x/a^2, the x-direction cosine of the normal to the surface of equilibrium; which however is impossible.

2. Let us now suppose the law of force to be a combination of those of the direct distance and the inverse square of the distance; that is

$$F = -\frac{\kappa^3}{r^2} - \mu r. \tag{13}$$

In this case

$$\Omega = \pi\rho\kappa^3(\alpha_o x^2+\beta_o y^2+\gamma_o z^2-\chi_o) + \tfrac{2}{3}\mu\pi\rho\, abc\,(x^2+y^2+z^2) \tag{14}$$

where

$$\left.\begin{aligned} \alpha_o &= abc\int_o^\infty \frac{d\lambda}{(a^2+\lambda)\Delta} \\ \beta_o &= abc\int_o^\infty \frac{d\lambda}{(b^2+\lambda)\Delta} \\ \gamma_o &= abc\int_o^\infty \frac{d\lambda}{(c^2+\lambda)\Delta} \\ \chi_o &= abc\int_o^\infty \frac{d\lambda}{\Delta} \end{aligned}\right\} \tag{15}$$

and $\Delta = \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$.

Substituting in (1), we have

$$x^2\left(\alpha_0\kappa^3+\tfrac{2}{3}\mu\,abc-\frac{\omega^2}{2\pi\rho}\right)+y^2\left(\beta_0\kappa^3+\tfrac{2}{3}\mu\,abc-\frac{\omega^2}{2\pi\rho}\right)$$
$$+z^2\,(\gamma_0\kappa^3+\tfrac{2}{3}\mu\,abc)=\text{const} \tag{16}$$

In order that this may coincide with the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

we must have

$$a^2\left(\alpha_0\kappa^3+\tfrac{2}{3}\mu\,abc-\frac{\omega^2}{2\pi\rho}\right)=b^2\left(\beta_0\kappa^3+\tfrac{2}{3}\mu\,abc-\frac{\omega^2}{2\pi\rho}\right)$$
$$=c^2\,(\gamma_0\kappa^3+\tfrac{2}{3}\mu\,abc) \tag{17}$$

(I) When $a=b$ these conditions reduce to

$$a^2\left(\alpha_0\kappa^3+\tfrac{2}{3}\mu abc-\frac{\omega^2}{2\pi\rho}\right)=c^2\,(\gamma_0\kappa^3+\tfrac{2}{3}\mu\,abc) \tag{18}$$

Substituting the values of α_0 and γ_0 from (15) we get

$$abc\,\kappa^3\int_0^\infty\left\{\frac{a^2}{a^2+\lambda}-\frac{c^2}{c^2+\lambda}\right\}\frac{d\lambda}{\Delta}+\tfrac{2}{3}\mu abc\,(a^2-c^2)=\frac{a^2\omega^2}{2\pi\rho}. \tag{19}$$

Since $a^2/(a^2+\lambda)$ is greater or less than $c^2/(c^2+\lambda)$ according as a is greater or less than c, it follows that the above condition can be fulfilled by any suitable value of ω for the planetary ellipsoid but not for the ovary form. Also the axis of rotation coincides with the smallest axis.

(II) When a is not equal to b, that is, when the ellipsoid is not one of revolution, we can put equations (17) in the form

$$a^2b^2\kappa^3\,(\alpha_0-\beta_0)+\gamma_0\kappa^3c^2\,(a^2-b^2)+\tfrac{2}{3}\mu\,abc^3\,(a^2-b^2)=0 \tag{20}$$

Substituting the values of α_0, β_0, γ_0,

$$-(a^2-b^2)\,\kappa^3\int_0^\infty\left\{\frac{a^2b^2}{(a^2+\lambda)\,(b^2+\lambda)}-\frac{c^2}{c^2+\lambda}\right\}\frac{d\lambda}{\Delta}+\tfrac{2}{3}\mu abc^3\,(a^2-b^2)$$
$$=0 \tag{21}$$

The factor (a^2-b^2) when equated to zero gives the ellipsoid discussed in (I). The other factor gives

$$-\kappa^3\int_0^\infty\left\{\frac{a^2\,b^2}{(a^2+\lambda)(b^2+\lambda)}-\frac{c^2}{c^2+\lambda}\right\}\frac{d\lambda}{\Delta}+\tfrac{2}{3}\mu abc^2=0$$

or $$-\kappa^3\int_0^\infty\{a^2\,b^2-(a^2+b^2+\lambda)\,c^2\}\,\frac{\lambda d\lambda}{\Delta}+\tfrac{2}{3}\mu abc^3=0 \tag{22}$$

which may be regarded as an equation determining c in terms of a, b. When $c=o$ every element of the integral is negative, and when $c^2=\dfrac{a^2\,b^2}{(a^2+b^2)}$ every element is positive. Hence there is some value of c

less than the smaller of the two semiaxes a and b for which the integral vanishes. Thus, an ellipsoid of three unequal axes is a possible form of equilibrium and further the axis of rotation is coincident with the smallest axis of the ellipsoid.

3. Poincaré has proved that, for the law of inverse square of the distance, if $\omega^2/2\pi\rho > 1$, there is no figure of equilibrium possible.* We have an analogous theorem for the law of direct distance. A necessary condition of equilibrium is that at every point of the free surface the resultant of the attraction and the centrifugal force should be directed towards the interior; otherwise a part would be detached. Let V be the potential of the attracting forces and r the distance from the axis, and let

$$U = V + \tfrac{1}{2}\omega^2 r^2. \tag{23}$$

The resultant outward normal force is $\frac{\delta U}{\delta n}$ and for equilibrium at every point of the free surface $\frac{\delta U}{\delta n}$ should be negative.

By Green's theorem

$$\iint \frac{\delta U}{\delta n} dS = \iint \nabla^2 U \, dx \, dy \, dz \tag{24}$$

where the first integral is taken over the surface and the second throughout the volume of the liquid. And

$$\nabla^2 U = \nabla^2 V + 2\omega^2. \tag{25}$$

When the force is that of the direct distance

$$\nabla^2 V = -3M\mu, \text{ M being the mass.} \tag{26}$$

$$\therefore \iint \frac{\delta U}{\delta n} dS = (2\omega^2 - 3M\mu) \times \text{volume} \tag{27}$$

and if $\omega^2 > \frac{3}{2}M\mu$, the left hand side is positive, which implies that at some points of the surface the resultant force is directed outwards and therefore the equilibrium is impossible.

* *Figures d'équilibre d'une masse fluide*, p. 11.

Parametric Coefficients in the Differential Geometry of Curves.

By Syamadas Mukhopadhyaya.

IV.—Expression of the co-ordinates of a point on a curve in an n-space as power series in s.

1. Let $\xi_1, \xi_2, \cdot, \xi_n$ be the n co-ordinates of a point P on a curve in an n-space referred to principal axes at any point O on the curve. Then on the hypothesis of the continuity of $\xi_1, \xi_2 \cdot, \xi_n$ and their derivates of any order with respect to s, the arc length from O to P, in the domain (OP) we can shew that

$$\xi_r = \underset{r}{\Delta}(r) \cdot \frac{s^r}{r!} + \underset{r}{\Delta}(r+1) \cdot \frac{s^{r+1}}{(r+1)!} + \cdot + \underset{r}{\Delta}(m)\frac{s^m}{m!} + \text{etc.},$$

$$\text{for } r=1, 2, 3, \cdot, n, \qquad (i)$$

$\Delta_r(m)$ having the same meaning as defined in Paper III.*

For if $X_1, X_2, \cdot, X_n$ be the n co-ordinates of P referred to any system of orthogonal axes and $x_1, x_2, \cdot, x_n$ the co-ordinates of O referred to the same system, then the direction cosines of the n principal axes at O are

$$\Delta_r(x_1), \Delta_r(x_2), \cdot, \Delta_r(x), (r=1, 2, \cdot, n)$$

where $\Delta_r(x_m)$ has the same meaning as defined in Paper III.**

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* $\Delta_r(m)$ stands for $[1, 2, \cdot, r \mid 1, 2, \cdot, r-1, m]$ divided by $[1, 2, \cdot, r]\,[1, 2, \cdot, r-1]$ and $[1, 2, \cdot, r \mid 1, 2, \cdot, r-1, m]$

$$\text{for} \begin{vmatrix} (1, 1), (1, 2), \cdot, (1, r-1), (1, m) \\ (2, 1), (2, 2), \cdot, (2, r-1), (2, m) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (r, 1), (r, 2) \cdot, (r, -r1), (r, m) \end{vmatrix}$$

$[1, 2, \cdot, r]^2$ stands for $[1, 2, \cdot, r \mid 1, 2, \cdot, r]$,

$[1, 2, \cdot, r-1]^2$ for $[1, 2, \cdot, r-1 \mid 1, 2, \cdot, r-1]$,

and (l, m) for $\sum \overset{(l)}{x_p}\ \overset{(m)}{x_p}$, $p=1, 2, \cdot, n$.

When $r=1$, $\Delta_r(m)$ is equal to $(1, m) \div (1, 1)^{\frac{1}{2}}$.

** $\Delta_r(x_m)$ stands for $[1, 2, \cdot, r \mid 1, 2, \cdot, r-1, x_m]$ divided by $[1, 2, \cdot, r]\,[1, 2, \cdot, r-1]$ and

Hence $\xi_r = \Sigma (X_p - x_p) \Delta_r (x_p), (p=1, 2, \cdot, n)$

$$= \Sigma \left(x_p' \cdot s + x_p'' \cdot \frac{s^2}{2!} + \cdot \cdot + x_p^{(m)} \frac{s^m}{m!} + \text{etc.}\right) \Delta_r (x_p)$$

$$= s \Sigma x'_p \Delta_r (x_p) + \frac{s^2}{2!} \Sigma x''_p \Delta_r (x_p) + \cdot + \frac{s^m}{m!} \Sigma x_p^{(m)} \Delta_r (x_p) + \text{etc.}$$

$$= s \Delta_r (1) + \frac{s^2}{2!} \Delta_r (2) + \cdot + \frac{s^m}{m!} \Delta_r (m) + \text{etc.}$$

by identify (a) paper III ‡.

$$= \frac{s^r}{r!} \Delta_r (r) + \frac{s^{r+1}}{(r+1)!} \Delta_r (r+1) + \cdot + \frac{s^m}{m!} \Delta_r (m) + \text{etc.},$$

as $\Delta_r (1), \Delta_r (2), \cdot, \Delta_r (r-1)$ all vanish from definition of Δ_r.

Equation (i), written in full, gives us

$$\xi_1 = \Delta_1 (1) \cdot s + \Delta_1 (2) \cdot \frac{s^2}{2!} + \cdot + \Delta_1 (m) \cdot \frac{s^m}{m!} + \text{etc.}$$

$$\xi_2 = \Delta_2 (2) \cdot \frac{s^2}{2!} + \Delta_2 (3) \cdot \frac{s^3}{3!} + \cdot + \Delta_2 (m) \cdot \frac{s^m}{m!} + \text{etc.}$$

....

$$\xi_r = \Delta_r (r) \cdot \frac{s^r}{r!} + \Delta_r (r+1) \cdot \frac{s^{r+1}}{(r+1)!} + \cdot + \Delta_r (m) \cdot \frac{s^m}{m!} + \text{etc.}$$

....

$$\xi_n = \Delta_n (n) \cdot \frac{s^n}{n!} + \Delta_n (n+1) \cdot \frac{s^{n+1}}{(n+1)!} + \cdot + \Delta_n (m) \cdot \frac{s^m}{m!} + \text{etc.} \quad (ii)$$

We have now to express the coefficients of $\xi_1, \xi_2, \cdot, \xi_n$ in terms of the $n-1$ principal curvatures at O, $u_1, u_2, \cdot, u_{n-1}$ and their derivates with respect to s

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$$[1, 2, \cdot, r \mid 1, 2, \cdot, r-1, x_m] \text{ for } \left| \begin{matrix} (1, 1), & (1, 2), & \cdot, & (1, r-1), & x_m^{(1)} \\ (2, 1). & (2, 2), & \cdot, & (2, r-1), & x_m^{(2)} \\ \cdots & \cdots & & \cdots & \cdots \\ (r, 1), & (r, 2), & \cdot . & (r, r-1), & x_m^{(n)} \end{matrix} \right|$$

‡ *Vide Bulletin* C. M. S. Vol. II, No. 2, Page 30.

We have from the definitions of the principal curvatures

$\Delta_2(2) = u_1,\ \Delta_3(3) = u_1 u_2,\ \Delta_4(4) = u_1 u_2 u_3, \cdot,$

$\Delta_r(r) = u_1 u_2 \cdot, u_{r-1}, \cdot, \Delta_n(n) = u_1 u_2, \cdot, u_{n-1},$

so that we can at once write down the first coefficient in each of the expressions (*ii*).

Evidently $\Delta_1(1) = (1, 1)^{\frac{1}{2}}$

$$= \left\{\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \cdot + \left(\frac{dx_n}{ds}\right)^2\right\}^{\frac{1}{2}} = 1.$$

The second coefficients in (ii) are obtained from the equations

$$\Delta_1(2) = 0$$

$$\Delta_2(3) = \Delta'_2(2) + u_1\,\Delta_1(2)$$

$$\Delta_3(4) = \Delta'_3(3) + u_2\,\Delta_2(3)$$

....

$$\Delta_r(r+1) = \Delta'_r(r) + u_{r-1}\,\Delta_{r-1}(r)$$

....

$$\Delta^n(n+1) = \Delta'_n(n) + u_{n-1}\,\Delta_{n-1}(n) \qquad (iii)$$

The first is evident since $\Delta_1(2) = (1, 2) \div (1, 1)^{\frac{1}{2}}$ $= (1, 2) = \frac{1}{2}(1, 1)' = 0$. The rest follow from the general formula (E_1), of paper III, *viz*,

$$\Delta'_r(p) = \Delta_{r+1}(p) \cdot u_r + \Delta_r(p+1) - \Delta_{r-1}(p) \cdot u_{r-1},$$

if we put r for p and observe that $\Delta_{r+1}(r) = 0$.

Similarly, the third coefficients in (ii) are obtained from

$$\Delta_1(3) = \Delta'_1(2) - u_1\,\Delta_2(2)$$

$$\Delta_2(4) = \Delta'_2(3) + u_1\,\Delta_1(3) - u_2\,\Delta_3(3)$$

....

$$\Delta_{n-1}(n+1) = \Delta'_{n-1}(n) + u_{n-2}\,\Delta_{n-2}(n) - u_{n-1}\,\Delta_n(n)$$

$$\Delta_n(n+2) = \Delta'_n(n+1) + u_{n-1}\,\Delta_{n-1}(n+1) \qquad (iv)$$

and the fourth coefficients in (ii) are obtained from

$$\Delta_1(4) = \Delta'_1(3) - u_1\,\Delta_2(3)$$

$$\Delta_2(5) = \Delta'_2(4) + u_1\,\Delta_1(4) - u_2\,\Delta_3(4)$$

... ...

$$\Delta_{n-1}(n+2) = \Delta'_{n-1}(n+1) + u_{n-2}\,\Delta_{n-2}(n+1) - u_{n-1}\,\Delta_n(n+1)$$

$$\Delta_n(n+3) = \Delta'_n(n+2) + u_{n-1}\,\Delta_{n-1}(n+2) \qquad (v)$$

The above sets of equations (iv) and (v) are deduced from general formula (E_1) by substituting suitable values for p and r.

By continuing the above methods the coefficients of the expansions of $\xi_1, \xi_2, \cdot, \xi_n$ in powers of s can be calculated with facility as far as we desire. G. B. Mathews in the Quarterly Journal of Mathematics Vol. XXVI (1893), pp. 27-30 has obtained by a different kind of analysis similar formulae for curves in three-dimensional space.

The following tables which have been constructed by the above methods give the coefficients up to those of the *ninth* power of s. For convenience of printing $\alpha, \beta, \gamma, \delta, \epsilon, \theta, \phi$, etc. have been substituted for the principal curvatures $u_1, u_2, u_3, u_4, u_5, u_6$, etc.

Poincare'—His Life and Work.

By A. C. Bose.

"The search for truth should be the goal of our activities, it is the only end that is worthy of it.........If we wish more and more to free man from material cares, it is that he may be able to employ the liberty obtained in the study and contemplation of truth."

So said Jules Henri Poincaré in his Introduction to his work 'the Value of Science' and in these few, but beautiful lines is embodied the motto of his life. But what is Truth said jesting Pilate and would not wait for an answer. And if Truth, whatever it is (opinions have differed regarding the meaning of truth, *e.g.* Ostwald has held "Truth is that which makes possible the prediction of the future") was to be the sole aim worth pursuing, Poincaré asked, could we hope to attain it? Such attainment might be doubtful. Is it then the case that our most legitimate and most imperative aspiration is at the same time the most vain? Or can we, despite all, approach Truth on some side? This was truly worthy of investigation and was investigated. What was the process and what the result? The frames in which Nature seemed enclosed, *viz.*, Time and Space, were examined. It was shown that Nature did not impose them upon us and that it was we who imposed them upon Nature, because we found them *convenient*. Was mathematical analysis whose principal object was the study of these empty frames, only a vain play of the mind? It gave the physicist, a convenient language but "was this not a mediocre service and was it not to be feared that this artificial language might be a veil interposed between reality and the eye of the physicist?" Poincaré answered boldly and decisively that it was not. "Without this language" he said, "most of the intimate analogies of things would have remained for ever unknown to us; and we should for ever have been ignorant of the internal Harmony of the world which is the only true objective Reality. The best expression of this Harmony is Law which is one of the most recent conquests of the Human Mind which we owe to Astronomy."

"Does the harmony," Poincaré asked, "the human intelligence thinks it discovers in nature exist outside of this intelligence?" The

answer came in no uncertain terms—"No, beyond doubt, a reality completely independent of the mind which conceives it, sees or feels it, is an impossibility. A world as exterior as that, even if it existed, would for us be for ever inaccessible. But what we call objective reality is, in the last analysis, what is common to many thinking beings and could be common to all; this common part would only be the harmony expressed by mathematical laws. It is this harmony which is the sole objective reality, the only truth we can attain. The universal Harmony of the world is the source of all beauty."

Poincaré's whole life was one continuous consecration to the search for this Harmony of the Universe and he approached Truth from many sides. As has been well said "his vision penetrated the Universe from the Electron to the Galaxy, from the instants of time to the Sweep of Space, from the fundamentals of thought to its most delicate propositions."

The *externals* of Poincaré's life like those of other great men who passed uneventful days in the "groves of the Academy" may be quickly summed up. He was born on April 29, 1854, at Nancy—the son of a physician highly respected. The Franco-German War of 1870-71 disturbed his studies but to be able to read the war news in the German papers, he learned German. His career at school was brilliant and in 1873 he unlike Hermite, who held all examinations in horror and who entered as sixty-eighth in his class a generation before Poincaré but whose future career was not less brilliant, passed highest into the Ecole Polytechnique, where he followed the courses in Mathematics, without taking a note and without the syllabus. He proceeded to the School of Mines in 1875 and was Nomme, March 26, 1879. On August 1, 1879, Poincaré won his Doctorate in the University of Paris and was appointed to teach in the Faculte des Sciences de Caen on December 1, 1879. In October, 1881, he was called to the University of Paris, where he taught till his death on July 17, 1912. At the early age of 32 he became a Member of the Academy of Sciences and in March 1908, was chosen a Member of the French Academy. Physically, as Charles Nordmann says in the Revue des Deus Mondes, September, 1912, "with his ruddy face, his beard turning a little grey and not always geometrically arranged, his shoulders bent as if under the ever present weight of his thoughts, the first impression of Henri Poincaré was one of the singular spirituality and imperious gentleness. But two traits were particularly characteristic of him, his voice deep and musical and remarkably animated when speaking of problems which greatly moved him, and his eyes, rather

small, often agitated by rapid movements under irregular eye-brows. In his eyes could be read the profound interior life which unceasingly animated his powerful brain (that "dome of thought" found by Sylvester on his visit to Poincaré in Paris). His glance was absent and kind, full of thought and penetration, his glasses scarcely veiling its depth and acuteness. His short-sightedness, poorly corrected by his glasses, added to his absent look and made one say of him "He is in the moon. Indeed he was often very far away." In Calcutta we possess a portrait of him in the rooms of the Mathematical Society an enlarged copy of what is given in Lebon's little volume on Poincaré in the series "Savants du Jour." Another picture of Poincaré in his younger days is contained in the American Journal of Mathematics. The change which took place with the process of years in the expression of the eyes is remarkable. It was a change from the eager, questioning and expectant look of a young explorer into the mysteries of the Universe to the startled look of the votary of science who has met with sights and known things not ordinarily given to mortals to see and know, a look of wonder and amazement not unmixed with one of triumph at having forced Nature to give up some of her secrets from the constitution of the rings of Saturn to the laws of the movement of electrons.

But not so quickly can be summed up the *internals* of Poincare's life—that life was so full, and so rich. To understand it the reader must accompany the writer, who would at best be a feeble guide, to the "silent arsenals where he slowly forged his weapons for the struggle against the unknown, to the workshop of the Mathematician, beneath the dome of the Astronomer or in the bare room which the Philosopher so richly furnishes with his meditations". The Method which pervaded his work, and which has been so well described by Emile Borel in La Revue du Mois of March 10, 1909, may be briefly mentioned here.

The Method of Poincaré.

"A knowledge of the Methods of a man of science is no less useful to the progress of science or toward his own glory than the discoveries themselves": *Laplace.*

"The Method of Poincaré" said Borel, "is essentially active and constructive. He approaches a question, acquaints himself with its present condition without being much concerned about its history, finds out immediately the new analytical formulas by which the question can be advanced, deduces hastily the essential results, and then passes to another question. After having finished the writing of a memoir, he

is sure to pause for a while, and to think out how the exposition could be improved; but he would not, for a single instance, indulge in the idea of devoting several days to didactic work. Those days could be better utilized in exploring new regions."

"Poincaré is a great constructor. He can exactly adapt his construction to the end he desires to attain. From this point of view his activity can be compared to that of men of action who crush all obstacles standing in their way. The difference is that the conquests of Poincaré are limited to the domain of thought."

"Let us now turn to the mathematical method of Poincaré. That method can be best described by saying that Poincaré is more a conqueror than a coloniser. He boldly advances into unexplored regions, and then leaving to others the work of organisation, he proceeds towards other region where his presence will be more necessary."

"Poincaré attaches little importance to conceptions which cannot be immediately realized in a concrete form. He is neither a dreamer nor an idealogist. His method of work is too active to leave room for any reflection which does not immediately lead to a concrete result."

"Thanks to that Method, he has been able to give to the world a scientific production which is the most considerable since Gauss and Cauchy, and which, not ceasing to grow in extent every year, will perhaps end by constituting the most important contribution that a mathematician has ever made to the intellectual patrimony of humanity."

Truly did Borel write in 1909 with insight born of intimate knowledge, for, although the great investigator closed his earthly labours after only three years, opinion is unanimous that he has made the most important contribution to the sum total of the intellectual achievements of humanity.

Mathematical Work.

In Ernest Lebon's volume on Poincaré (Savants du Jour, Gauthier Villars, Paris, 1909) is given a list of 436 different publications, 98 of which relate to Pure analysis. In the Theory of Numbers he introduced the new and fruitful idea of arithmetical invariants and his discoveries in the general theory of functions were remarkable and numerous. As Halsted says "his earliest publication was in 1878 (*i.e.* shortly before he received the doctorate at the University of Paris) which was not important. Afterwards came an essay submitted in competition for the Grand Prix in 1880 but it did not win. Suddenly there came a change, a striking fire, a bursting forth, in February, 1881." This relates to his sudden discovery, at

the moment Poincaré had put his foot on the step of an omnibus, that the transformations he had used to define Fuchsian functions (of which more anon) were identical with those of Non-Euclidean Geometry. In three short years (1879-81) Poincaré had contributed to the pages of the Comptes Rendus "his epoch-making results on the uniformisation of algebraic curves and the solution of Linear Differential equations which at the early age of 27 gave him a recognised position in the front rank of mathematicians." His career, thus, in the earliest stage was meteoric but the meteor did not lose itself in space but by some curious process, yet to be discovered by astronomers, was transformed into a brilliant Sun, which shed, for a generation, such a benignant light on many a world of thought and endeavour.

(*a*) Differential Equations, their Genesis and importance in Physics.

Sophus Lie has pointed out that "among all the disciplines, the theory of differential equations is the most important. It furnishes the explanation of all those elementary manifestations of Nature which involve time." These remarks require amplification and I cannot do better than quote from a French writer on the genesis of such equations:—

"Newton was the first to show that the state of a moving system, or more generally that of the universe, depends only on its immediately preceding state, and that all the changes in nature take place in a continuous manner. A law, then, is only the necessary relation between the present state of the world and that immediately preceding. It is a consequence of this, that in place of studying directly a succession of events we may limit ourselves to considering the manner in which two successive phenomena occur; in other words we may express our succession by a differential equation. All natural laws which have been discovered are only differential equations. Looking at it slightly differently, such equations have been possible in physics because the greater part of physical phenomena may be analysed as the succession of a great number of elementary events, "infinitesimals," all similar. The knowledge of this elementary fact allows us to construct the differential equation and we have then to use only a method of summation in order to deduce an observable and verifiable complex phenomenon. This mathematical operation of summation is called the "integration" of the differential equation. In the greater number of cases this integration is impossible and perhaps all progress in physics depends on perfecting the process of integration. That

was the principal work of Poincaré in Mathematics. And in that line his work was amazing, especially in the development of those now famous functions, the simplest of which are now called Fuchsians after the German Mathematician Fuchs, whose work had been of aid to Poincaré."

As put by Professor J. B. Shaw of the University of Illinois, "to Poincaré the world of relations was as real as the world of phenomena, and so far as we know the real relations, in whatever language we express them, just so far we know the actual world, the objective world."

The value of differential equations consists in that it is the language which expresses "certain persistent relations between phenomena and is thus real and is the replica of an objective reality."

In recent years this conception of the role of differential equations in the domain of physics has been broken in upon by other conceptions, each representating a relation between states. The truth of the old adage "Natura non fecit saltus" has been questioned. The Integro-differential equation of Volterra expresses the relation that the present state of the world is due to *all* the preceding states; the Difference equation puts the relation that the states follow one another *abruptly*; and the Integro-difference equation the relation that they depend on all preceding states *discontinuously*. It has been well said that "We are thus witnesses of an evolution in Science and Mathematics from the continuous to the discontinuous. In Mathematics it has produced the function defined over a range, rather than a line—a chaos as it were of elements, and the calculable numbers of Borel." In physics, it has produced the electron, the magneton and the Theory of Quanta, about which shortly before his death Poincaré said.

"A physical system is capable of only a finite number of distinct states; it abruptly jumps from one state to another without passing through the intermediate states."

This shows a remarkable change of view. Is Newton to be abandoned for later lights? Perhaps not: We require only to write the Newtonian mechanics in a different language. Compare the Lamarkian and Darwinian theories of slow and imperceptible evolution with the sudden and discontinuous mutation of the Dutch naturalist De Vries. The new evidence brought forward by the latter has not demolished the older theory; its greater signification remains. Paradoxical as it may seem, Poincaré puts it right when he says "this is right and the other is not wrong. They are in harmony; only the language varies—both set forth certain true relations." "Similarly the Theory of Quanta,

it is probable, will not prevent, the greater part, if not all, of physical phenomena, from being capable of representation by differential equations." The progress hitherto made in physical discoveries is the surest guarantee that this would be so, and the discoveries made by Poincaré in the Theory of differential equations as mentioned below will retain their essential value.

(b) The Theory of functions :—Fuchsians and Fuchsian groups.

About the middle of the last century mathematicians broke new ground in endeavouring to determine the nature of the function defined by a differential equation from the equation itself and not from any analytical expression obtained by solving it. They at first studied the properties of the integrals in the vicinity of a given point and found that the nature of the integrals at singular points and that at ordinary points differed. Fuchs gave the development in series near a singular point of the integrals for the particular case of linear differential equations. Poincaré did the same for the case when the equations are not linear and also for partial differential equations of the first order. The developments for ordinary points were given by Cauchy and Madame Kowalevsky. Poincaré, by the introduction of new transcendents, succeeded in expressing the integrals by developments which were always convergent and not limited to particular points. Confining himself to linear differential equations with rational algebraic co-efficients, he was able to integrate them by the use of functions which were named by him Fuchsians. He divided these equations into "families." If the integral of such an equation be subjected to a certain transformation, the result will be the integral of an equation belonging to the same family. The new transcendents thus introduced have a great analogy to elliptic functions; while the region of the latter may be divided into parallelograms, each representing a group, the former may be divided into curvilinear polygons, so that the knowledge of the function inside of one polygon, carried with it the knowledge of it inside the others. Thus, Poincaré arrived at what he called Fuchsian Groups. The extension to non-linear equations of the method applied to linear equations was attempted both by Fuchs and Poincaré.

The mode of integration which gives the properties of equations, from the point of view of the Theory of functions, was not found sufficient in the application of differential equations to applied mathetics. If we consider the function as defining a plane curve, then the form of the curve could not be judged from the above procedure. Poincaré carried on investigations to construct the curves defined by differential equations.

We thus see that by the use of Fuchsian functions, Poincaré could solve differential equations, could express the co-ordinates of algebraic curves as Fuchsian functions of a parameter and could solve algebraic equations of any order. He utilized the Fuchsians in the theory of Arithmetic invariants which opened up immense possibilities for the development of the Theory of Numbers. The flash of his genius had illumined the bridge between the two viz., the theory of Groups of linear substitutions. Truly then, as his colleague M. Humbert of the Académie des Sciences said, "Poincaré handed us the keys of the world of algebra."

(c) Role of Intuition in Mathematical discoveries.

In his most illuminating essay on Mathematical creation (Science and Method) Poincaré has let us see the inside of the workshop of a truly great mathematician and has disclosed a chapter of psychology which would repay a careful study by all who are anxious to do constructive work. It is time, he says, "to penetrate deeper and see what goes on in the very soul of the mathematician. For this, I believe, I can do best by recalling memories of my own......"

"For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions. I was then very ignorant; every day I seated myself at my work table, stayed an hour or two, tried a great number of combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning, I had established the existence of a class of Fuchsian functions, those which come from the hypergeometric series; I had only to write out the results, which took but a few hours."

"Then I wanted to represent these functions by the quotient of two series; this idea was perfectly conscious and deliberate, the analogy with elliptic functions guided me. I asked myself what properties these series must have if they existed, and I succeeded without difficulty in forming the series I have called theta-Fuchsian."

"Just at this time I left Caen, where I was then living, to go on a geologic excursion under the auspices of the School of mines. The changes of travel made me forget my mathematical work. Having reached Coutances, we entered an omnibus to go to some place or other. At the moment when I put my foot on the step the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations I had used to define the

Fuchsian functions were identical with those of non-Euclidean geometry. On my return to Caen, for conscience' sake, I verified the result at my leisure."

"Then I turned my attention to the study of some arithmetical questions apparently without much success and without a suspicion of any connection with my preceding researches. Disgusted with my failure, I went to spend a few days at the sea-side, and thought of something else. One morning, walking on the bluff, the idea came to me, with just the same characteristics of brevity, suddenness and immediate certainty, that the arithmetic transformations of indeterminate ternary quadratic forms were identical with those of non-Euclidean geometry."

"Most striking at first is this appearance of sudden illumination, a manifest sign of long, unconscious prior work. The role of this unconscious work in Mathematical invention appears to me incontestable, and traces of it would be found in other cases where it is less evident. Often when one works at a hard question, nothing good is accomplished at the first attack. Then one takes a rest, longer or shorter, and sits down anew to the work. During the first half-hour, as before, nothing is found, and then all of a sudden the decisive idea presents itself to the mind. It might be said that the conscious work has been more fruitful, because it has been interrupted and the rest has given back to the mind its force and freshness. But it is more probable that this rest has been filled out with unconscious work and that the result of this work has afterwards revealed itself to the geometer just as in the cases I have cited, only the revelation, instead of coming during a walk or a journey, has happened during a period of conscious work, but independently of this work which plays at most a role of excitant, as if it were the goad stimulating the results already reached during rest, but remaining unconscious to assume the conscious form."

"Such are the realities; now for the thoughts they force upon us. The unconscious, or, as we say, the subliminal self plays an important role in Mathematical creation; this follows from what we have said. But usually the subliminal self is considered as purely automatic. Now Mathematical work is not simply mechanical; it could not be done by a machine, however perfect. It is not merely a question of applying rules, of making the most combinations possible according to certain fixed laws. The combinations so obtained would be exceedingly numerous, useless and cumbersome. The true work of the inventor consists in choosing among these combinations so as to eliminate the useless ones or rather to avoid the trouble of making them, and the

rules which must guide this choice are extremely fine and delicate. It is almost impossible to state them precisely; they are felt rather than formulated. Under these conditions, how imagine a sieve capable of applying them mechanically?"

"A first hypothesis now presents itself: the subliminal self is in no way inferior to the conscious self; it is not purely automatic; it is capable of discernment; it has tact, delicacy; it knows how to choose, to divine. What do I say? It knows better how to divine than the conscious self, since it succeeds where that has failed. In a word, is not the subliminal self superior to the conscious self?" * * *

"It may be surprising to see emotional sensibility invoked *a propos* of Mathematical demonstrations which, it would seem, can interest only the intellect. This would be to forget the feeling of Mathematical beauty, of the harmony of numbers and forms, of Geometric elegance. This is a true esthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility."

"Now, what are the Mathematic entities to which we attribute this character of beauty and elegance, and which are capable of developing in us a sort of esthetic emotion? They are those whose elements are harmoniously disposed so that the mind without effort can embrace their totality while realizing the details. This harmony is at once a satisfaction of our esthetic needs and an aid to the mind, sustaining and guiding. And at the same time, in putting under our eyes a well-ordered whole, it makes us foresee a Mathematical law. Now, as we have said above, the only Mathematical facts worthy of fixing our attention and capable of being useful are those which can teach us a Mathematical law. So that we reach the following conclusion: The useful combinations are precisely the most beautiful, I mean those best able to charm this special sensibility that all Mathematicians know but of which the profane are so ignorant as often to be tempted to smile at it."

"What happens then? Among the great numbers of combinations blindly formed by the subliminal self, almost all are without interest and without utility; but just for that reason they are also without effect upon the esthetic sensibility. Consciousness will never know them; only certain ones are harmonious, and, consequently, at once useful and beautiful. They will be capable of touching this special sensibility of the geometer of which I have just spoken, and which, once aroused, will call our attention to them, and thus give them occasion to become conscious."

To the Research student the above is of incalculable value. It traces not only the genesis of a certain class of powerful functions, but shows also how they were extended to other regions. We ask, what is the mechanism of the extensions, how can we devise new formulæ? how make new constructions? Poincaré points out that the answer is to be found in Psychology. The possibility lies in the power of the mind—Poincaré calls Intuition. It is that power "which enables us to perceive the plan of the whole, to seize the unity in the matter at hand." Those who possess the kind of insight which reveals hidden relations, may hope to become investigators. Too prolonged adherence to the methods of logic is likely to lead to sterility. In Mathematics at least both Intuition and Logic are indispensable, "one furnishes the architect's plan of the structure and the other bolts it and cements it together." Intuition is the sole instrument of creation, logic of certitude. "Intuition is the direct appreciation of relationship between the objects of thought which unite them into a complete structure unitary in character and harmonious in form."

There are types of intuition. There is the Visualist type and we recollect of Faraday and his lines of force, of Kelvin and his models of Ether. "Bertrand and Hermite, although schoolmates and educated on the same method represented different types—the Visual and the Symbolic. Bertrand, when speaking, was always in motion trying to "paint his ideas." "Hermite seemed to flee the world, his ideas were not of the visible kind. Weierstrass thought in artificial symbols, Riemann in pictures and Geometric constructions." Poincaré combined in him three types—the Audile (he remembered sounds well), the Visual and the Symbolic."

In addition to the Fuchsians and theta-Fuchsians, Poincaré made other notable contributions to the general Theory of Functions and the Theory of Groups. He studied and wrote on the properties of Abelian functions. Functions of a Complex variable had been studied since Riemann by Weierstrass and Mittag-Leffler. Poincaré dealt with many interesting problems arising out of the work of the former. He first gave an example of a Fuchsian having an infinite number of singular points but no singular lines and no *isolated* singular point. Uniform functions of two variables unaltered by certain linear transformations were dealt with by him and he called them "Hyper-fuchsians." On the lines of Weierstrass he studied functions uniform only in lacunary spaces and disclosed the mechanism of the generation of such functions. Remarkable was his proof that there was no means of generating them by which the lacunæ could be avoided. Writing in Vol. XXII of the

Acta Mathematica on the now famous function of Weierstrass (*Vide* Dr. Ganesh Prasad's Differential Calculus, pages 140-143), he showed how the exercise of the Visual type of intuition divorced from Logic might lead to error. True was his estimate of the role of the Theory of Functions when he said "by the discovery of this striking example Weierstrass has given us a useful reminder and has taught us better to appreciate the faultless and purely arithmetical methods with which he, more than any-one, has enriched our Science." To Poincaré is due the general classification of Automorphic functions and the Theory of Elliptic Modular Functions, started by Eisenstein's Memoir in 1847, benefited by his contributions made since 1881.

(The student curious to know Poincaré's contributions on the Theory of functions might certainly turn to the Acta Mathematica usefully, to the Comptes Rendus, and to Liouville. The first, third, fourth and fifth volumes of the Acta contain his papers on Automorphic functions).

Tchebycheff of the St. Petersburgh University had established on very elementary considerations in a celebrated memoir (1850) the existence of limits within which the sum of the logarithms of the primes, P, inferior to a given number X must be comprised. Poincaré's papers with Sylvester's and Hadamard's are the latest researches in the line.

In his celebrated Lectures delivered on April 22-28, 1909 at Göttingen, Poincaré treated "a wide range of interesting subjects in a masterly and illuminating way." The topics were

(1) The Fredholm integral equations.

(2) The application of the Theory of integral equations to fluid motions and to Hertzian Waves.

(3) The reduction of Abelian integrals and the Theory of Fuchsian functions.

(4) Transfinite numbers.

(5) The New Mechanics.

(1) and (2) :—" Struck by the fact that integral equations of a certain type occur very frequently in the linear problems of Mathematical physics, the Swedish Mathematical physicist, Ivar Fredholm undertook their study and his fundamental Memoir was published in 1903 in Vol. XXVII of the Acta and dedicated to the Memory of Abel on the occasion of the Centenary of his birth. Hilbert in 1909, Koch in 1910, Bateman and Hahn in 1911, Heywood and Frechet jointly in

1912 have carried on researches on this new arrival in the province of Mathematics. Poincaré threw much light on the subject in his Göttingen lecture. (Those who are interested in Integral equations, may refer to Boecher's Tract on the same in the Cambridge series, Dr. Ganesh Prasad's Review of the Tract in the Bulletin to the Calcutta Mathematical Society (p.p. 153-154), and Professor E. H. Moore's Vice-Presidential Address on the Foundations of the Theory of Linear Integral equations before the American Society for the Advancement of Science, 1911—reprinted in the Bulletin of the American Mathematical Society, Volume 18, p.p. 334-362. Some interesting facts are contained also in Professor G. B. Mathew's Review of Professor Lalesco's and Frechet and Heywood's books on Integral Equations, in Nature, July 18, 1912).

(4). "The lecture on Transfinite numbers develops Poincaré's attitude towards some of the subtleties in the controversial field of Mathematics brought into being by Georg Cantor. Poincaré concluded that the contradiction between Richard's proof that the Continuum was denumerable and Cantor's proof that it was not so, was not a real one. And he passed on to point out how the demonstration that "every algebraic equation has a root" needed to be re-stated.

(5). This was a popular lecture given in French, and Poincaré's views are contained in the chapter on "the new Mechanics and Astronomy" in his work on "Science and Method."

The above is but a meagre account of the contributions of this great son of France to the progress of mathematics. Poincaré was modest when he said "My daily mathematical studies—how shall I express myself—are esoteric and many of my hearers would revere them more from afar than close to." By such modesty "he only makes us pardon his genius." A competent authority has said "it would require a dozen years of preliminary mathematical study for the curious reader to be able to *know* them (Poincaré's mathematical researches) and if he were familiar with the elements such as he would get in the ordinary college course he might take a *glance* at them." But, to sum up, we may say with Professor Shaw of the Illinois University that "the Fuchsian functions opened an immense field of investigation. Poincaré created a type of arithmetic invariants expressible as series or definite integrals, which opened a new field in the Theory of numbers. His investigations of ordinary differential equations which are not linear such as those in dynamics and the problem of n bodies, created an extensive class of new functions which are yet without special names, as well as suggesting the existence of classes of func-

tions for which we have as yet, no means of expression. The investigations of Asymptotic Expansions opened paths to dizzy heights (refers to Poincaré's researches on the stability of our universe). Fundamental functions in partial differential equations also open a region now under development. The most marvellous of his creations rise from the general field of differential equations. We might cite further his researches in Analysis Situs—the realm of the invariants of a battered continuity. His double residues and studies in functions of many real variables are creations from which will spring a noble progeny. Even the lectures in which he presented the results of others scintillate with original thoughts" (Popular Science Monthly for March, 1913).

It has been said that "the Modern Theory of functions is the stateliest of all the Pure creations of the human intellect." And none contributed more largely to the rearing of that noble structure than Poincaré. His notable characteristic was his power of generalization. As Klein has remarked, the great need of the day is the discovery and elaboration of some unifying principle in the different branches of mathematics. Poincaré's genius was nowhere better employed than in this grand task. We have seen the display of this power in his discovery of the Fuchsians and Zetafuchsians and their application to such branches as the Theory of Numbers and the Theory of Curves. He studied Continuous Groups and applied them to hyper-complex numbers and then applied the latter to the periods of Abelian integrals and the algebraic integration of differential equations of certain types. It is largely due to him that the Theory of the Solution of systems of an infinite number of linear equations with an infinite number of unknowns received an adequate treatment. To him we owe the establishment of definite criteria of convergence in reference to the infinite determinants employed by Hill with so much success in Astronomy. His endeavours knew no bounds. He applied the Kinetic Theory of gases and the Theory of radiant matter to the Milky Way itself, suggesting that our system was probably a speck in a spiral nebula. He analysed *mathematically* the rings of Saturn into a swarm of sattelites which the spectroscope confirmed; this reminds us of the analogous cases of the discovery of Neptune and Conical refraction, in which the power of Pure mathematics to help in discoveries was demonstrated. Thus in his life and work, we find a realization of Poincaré's conception that "Mathematics has a triple end. It should furnish an instrument for the study of nature. Furthermore, it has a philosophic end and an end esthetic." And Poincaré held that mathematics deserves to be cultivated for its own sake and that the

theories not admitting of application to physics deserve to be studied as well as others.

Astronomical Work.

Poincaré was undoubtedly one of the greatest of Astronomers and he brought to bear on the solution of some of the outstanding problems of celestial mechanics, his incomparable powers of mathematical analysis. In early life he won half the prize offered by King Oscar II of Sweden for the solution of a question in reference to the problem of 3 bodies. His paper was published in the Acta Mathematica in 1890. It is an interesting fact, as pointed out in 1904, in the Bibliotheca Mathematica by Enestrom that the copy of the memoir for which the prize was actually awarded contained a serious error, and that the published article was really prepared for the press after the prize had been awarded.

Poincaré's researches on the stability of our universe deservedly rank very high. This study had been the fundamental problem of Astronomy for ten centuries. What was to be the cumulative effect of the planetary disturbances through the ages? that was the grand problem to be solved. Was there or was there not the probability of a catastrophe? Was the simple harmony of the world of Kepler no longer to be real? Newton's mind was obviously disturbed by these thoughts when he wrote in his Optics regarding the planetary inequalities "they probably will become so great in the long course of time that finally the system will have to be put in order by its Creator."

Laplace believed in 1772 that he was able to demonstrate that these fears were groundless. He showed that the secular inequalities of the planetary elements compensated themselves periodically at the end of a sufficiently long period and the terms of the first order of perturbations would disappear in the calculations. He thus believed that he had practically established the stability of our Solar system at least for thousands of secular periods.

Lagrange and Poisson extended the results of Laplace and the indefinite stability of the planetary elements seemed to be beyond doubt. "God, as Newton feared, would not be obliged to retouch his work."

Poincaré now attacked with his mighty mathematical tools, the problem of three bodies. As put popularly by a French writer "the problem could be solved only by the method of successive approximations. In the equations of Laplace and his followers, the co-ordinates of the planets were developed in a series whose terms were arranged in powers of the masses. Poincaré showed that he could not thus

obtain an indefinite approximation and that the convergence of the series had been assumed without proof and it was probable that in the terms of the higher order, the time enters not only with the sine and cosine, which would lead to periodic compensations of the irregularities but also *outside of the trigonometric functions*, so that certain of the terms at first negligible, might possibly increase indefinitely with the time. Here with one blow Poincaré reduced to naught the conclusions of Laplace and his successors."

Poincaré was not content by being destructive only. His constructive genius soon led him to devise new means—a brilliant series of new theorems of great generality by which he could express in every case co-ordinates of the planets in a purely trigonometric series. "The rigorous proof of the stability now depended only on whether the new series would be convergent." This was the crux of the problem Mathematicians before him had supposed a trigonometric series to be absolutely convergent. It was the credit of Poincaré that he showed that this opinion, " though classic was erroneous" and that even when we have represented the co-ordinates by a convergent series which is not very different from that employed by Laplace, we shall not have demonstrated the stability of the Solar system. Poincaré applied even the Theory of Probabilities to show that the stability or instability of our universe has never been demonstrated but that "if probability is measured by continuous functions only, the universe is most probably stable". Poincaré's Les Methodes Nouvelles de la mecanique celeste, "the crowning of three centuries of incessant research" an admirer of Poincaré has said, will take its place by the side of the imperishable Principia of Newton. (The critical reader, however, may profitably read Hill's article on the Convergence of the Series used in the subject of perturbations in pp. 93—99 of Vol. II, 1896, of the Bulletin of the American Mathematical Society, in which Hill contests Poincaré's conclusions).

Next in importance to the problem of 3 bodies is the problem of the shape of a star which reduces itself to that of a fluid mass rotating subject to various forces. Poincaré's researches mark "an epoch in the study of the subject" as the late Sir George Darwin said when he handed over to Poincaré the gold medal of the Royal Astronomical Society of London. Formerly only two figures of equilibrium for rotating fluids were known, *viz.* the Ellipsoid of revolution and Jacobi's Ellipsoid with three unequal axes. Poincaré tried to establish an infinite number of others which are stable and shaped like pears. These Apiodes seem to have an important place in nature as shown by the study of

certain nebulæ and double stars, which enables us to form some idea of the "mechanism of the bipartition, somewhat analogous to that of organic cells, which may have given birth to a great number of binary systems and which successively separated the earth from the Sun and then the Moon from the earth".

Poincaré showed that no form of equilibrium was stable when the velocity of rotation exceeded a certain limit. He applied this to the ring system of Saturn. Maxwell held that the rings could not be solid and if fluid, their density could not exceed $\frac{1}{100}$ths of the density of Saturn. Poincaré proved that if the rings were fluid they could not be stable unless their density was greater than $\frac{1}{80}$th of the density of Saturn. He concluded that the only alternative was to suppose that "they were formed of a multitude of small satellites, gravitating independently. We know how "spectrum analysis has subsequently proved this marvellous deduction of this mathematical genius."

As in the domain of mathematics, so also in that of celestial mechanics it is idle to endeavour to summarise the work of Poincaré within the compass of a short paper. As Professor H. H. Turner has said in his Address before the British Association in 1911, in reviewing Poincaré's Theory des Marees (Tome III of the Leçons de Mec. Celeste).

"Everything that M. Poincaré writes is delightful and the only fault to be found is that there is no index. But after all a *romance* needs no index and there is much of the charms of romance in these pages. Even the mathematical formulæ have by some logical process lost their long tails of crabbed numbers (and my goodness! what numerical monstrosities the tides can produce) and dance and transform themselves like fairy things at the hands of the wizard writer."

Characteristic was Poincaré's decision as expressed in this work about the practical suggestions (re-accounting numerically for the ocean tides) made by Mr. R. A. Harris of the U. S. Coast Survey. Poincaré while thinking favourably of Harris's central idea did not trust his deductions from *experiments* (one of which was shown to be wrong) and expressed his opinion that we could get closer to the facts by *actual integration* over the critical areas. The labour would be enormous, but it could be faced and a suitable method was furnished by the discoveries of Fredholm.

The student of celestial mechanics who is anxious to know and compare the results obtained by other workers in the field, would do well to study Hill 'smasterly summing up of the progress of celestial

mechanics since the middle of the 19th century (pp. 125-136 of Volume II of the Bulletin of the American Mathematical Society, 1896). Professor Hill concludes by saying :—

We owe much to M Poincaré for having commenced the attack on this class of questions (viz., 1. Periodic Solutions. 2. Asymptotic Solutions. 3. Development of the integrals of planetary motion according to the powers of a small parameter) but the mist which overhangs them is not altogether dispelled ; there is room for further investigation.

Mathematical Physics.

In early times the mathematician and the physicist were one and the same man but with the increasing complexity of both the sciences specialization became necessary and the Pure Mathematician became distinct from the Pure physicist, the separation becoming gradually most marked about the end of the 19th century. When once the utility of Bessel's functions in connection with applied mathematics was being discussed, a great Pure Mathematician who was present said "Yes, Bessel's functions are very beautiful functions *in spite* of their having practical applications." Educationists like Professor Perry have questioned whether Pure Mathematics should be cultivated divorced from its physical applications. Dr. Hobson has shown in his Address in 1912 to the Mathematical and Physical Society of the University College, London ("Mathematics from the point of view of the Mathematician and of the Physicist". Cambridge University Press—Price 1/s.), what the most rational answer to this much controverted question is and the writer would advise every student of mathematical physics a careful perusal of this pamphlet. As Dr. Hobson rightly puts it, "Mathematics can in the long run be developed to the highest degree of perfection, not only from the point of view of specialists within its own domain, but also as constituting an essential component of the intellectual life and stock of ideas of the world, only on the condition that it is allowed full freedom of self-expression. The utilitarian notion, in this connection as in so many others, has the fatal limitation that it attempts to assign limits to what is, or may in the future become, useful, in accordance with a more or less arbitrarily restricted standard of what constitutes utility."

Although Poincaré's conquests lay mostly in the region of Pure thought, his restless activity found proper exercise in the field of mathematical physics. Capillarity, Elasticity, Newtonian Potential, the analytical Theory of Heat, Thermodynamics, Diffraction and

dispersion of light, Electric Oscillations, Maxwell's Electro-magnetic Theory, Hertzian Waves, Lorentz's Theory, Zeeman's Experiments and the Theory of Relativity all had the benefit of his masterly treatment.

Poincaré criticized Maxwell's Electro-magnetic Theory in Part I of his Electricité et Optique and his criticisms have been reviewed in a friendly spirit in Nature, pages 296-299, of 1891 and pages 367-372, of 1892. And the somewhat drastic criticism levelled at Poincaré by the late Professor Fitzgerald ("Poincaré and Maxwell," pages 532-533, Nature, 1892) bespeaks the attitude of the leading physicists of the time in the United Kingdom, towards the excursions of this great French mathematician in fields which they considered peculiarly their own.

Poincaré's work on Thermo-dynamics exposed him to the adverse criticisms of P. G. Tait, an incomparable debater (Nature Vol. XV, for 1891-92). In Tait's Life and Work recently published by the Cambridge Press (pages 273-276) is given an account of this controversy. The impression is left that the great mathematician, in his whole-hearted devotion to Analysis, forgot to attach due importance to the work of great experimenters and physicists and forgot to do honour even to his illustrious countryman Sadi Carnot and Tait was perhaps justified in saying :—

"Monsieur Poincaré not only ranks very high indeed among pure mathematicians but has done much excellent and singularly original work in applied mathematics. All the more, therefore, should he be warned to bear in mind the words of Shakespeare :—

"Oh, it is excellent to have a giant's strength ; but it is tyrannous to use it like a giant."

Philosophy of Science.

It has been well said that "with the sudden death of Henri Poincaré came a great sadness to all lovers of *idealism.*" Speaking of his mathematical achievements Painlevé said "yet this was only part of his activity : Geodesy, Cosmogeny, Astronomy, Philosophy of Science, he included them all, penetrated all, explored all." Unable to escape that "attraction which has forced all the great workers in the exact sciences from Democritus to D'Alembert toward the end of their lives, to reflect upon the primordial mysteries of the strange universe wherein our ephemeral thoughts live and die," Poincaré has given to the world, in his Science and Hypothesis, Science and Method and Value of Science, his thoughts on the fundamental problems of science. These philo-

sophical writings of Poincaré have deeply impressed the thinking world and are probably the most widely read of his productions. These have been translated into six different languages, and with the publication in 1913 of Halsted's translation into English of all the three works under the title of "the Foundations of Science" by the Science Press of New York, the necessity has ceased of an exposition in English of Poincaré's main standpoints, regarding Number and Magnitude, Space, Force and Nature. But a few extracts from Poincaré's own preface to Halsted's Translation may not be out of place here.

The contrast between the Anglo-Saxon spirit and Latin spirit in the study of Nature.

"The Latins seek in general to put their thought in mathematical form ; the English prefer to express it by a material representation."

"Both doubtless rely only on experience for knowing the world ; when they happen to go beyond this, they consider their fore-knowledge as only provisional, and they hasten to ask its definitive confirmation from nature herself."

"For a Latin, truth can be expressed only by equations ; it must obey laws simple, logical, symmetric and fitted to satisfy minds in love with mathematical elegance."

"The Anglo-Saxon to depict a phenomenon will first be engrossed in making a *model*, and he will make it with common materials, such as our crude, unaided senses show us them. He also makes a hypothesis ; he assumes implicitly that nature, in her finest elements, is the same as in the complicated aggregates which alone are within the reach of our senses. He concludes from the body to the atom."

"Both therefore make hypotheses, and this indeed is necessary, since no scientist has ever been able to get on without them. The essential thing is never to make them unconsciously."

"From another point of view, however, the two conceptions are very unlike, and if all must be said, they are very unlike because of a common fault."

"The English wish to make the world out of what we see. I mean what we see with the unaided eye, not the microscope nor that still more subtile microscope—the human head guided by scientific induction."

" The Latin wants to make it out of formulas, but these formulas are still the quintessenced expression of what we see. In a word, both would make the unknown out of the known, and their excuse is that there is no way of doing otherwise."

"And yet is this legitimate, if the unknown be simple and the known, the complex?"

"Is not each great advance accomplished precisely the day some one has discovered under the complex aggregate shown by our senses something far more simple, not even resembling it, as when Newton replaced Kepler's three laws by the single law of gravitation, which was something simpler, equivalent, yet unlike?"

"One is justified in asking if we are not on the eve of just such a revolution or one even more important. Matter seems on the point of losing its mass, its solidest attribute, and resolving itself into electrons. Mechanics must then give place to a broader conception which will explain it, but which it will not explain."

"So it was in vain the attempt was made in England to construct the ether by material models, or in France to apply to it the law of dynamic."

"The ether it is, the unknown, which explains matter, the known; matter is incapable of explaining the ether."

Death.

Poincaré died on July 17, 1912. In May of the same year he came to London and delivered a course of lectures before the University on (1) the Logic of the Infinite, (2) Time and Space, (3) Arithmetical invariants and (4) the Theory of Radiation (for an excellent Summary of these lectures, Nature of May 16, 1912 may be consulted). He was eagerly expected at the Congress of the World's Mathematicians held at Cambridge in August, 1912. Keen was the disappointment when the news was received that this great mathematician was no more. Sir George Darwin, President, spoke as follows :—

"Up-to a few weeks ago there was one man who alone of all mathematicians might have occupied the place which I hold without misgivings as to his fitness; I mean Henri Poincaré. It was at Rome just four years ago that the first dark shadow fell on us of that illness which has now terminated so fatally. You all remember the dismay which fell on us when the word passed from man to man "Poincaré is ill." We had hoped that we might again have heard from his mouth some such luminous address as that which he gave at Rome; but it was not to be, and the loss of France in his death affects the whole world."

"It was in 1900 that, as president of the Royal Astronomical Society, I had the privilege of handing to Poincaré the medal of the Society,

and I then attempted to give an appreciation of his work on the theory of the tides, on the figures of equilibrium of rotating fluid and on the problem of three bodies. Again in the preface to the third volume of my collected papers I ventured to describe him as my patron Saint as regards the papers contained in that volume. It brings vividly home to me how great a man he was when I reflect that to one incompetent to appreciate fully one-half of his work, yet he appears as a star of the first magnitude."

Professor E. B. Elliot called Poincaré the Prince of Analysts. Bertrand Russell, that philosophic mathematician, praised his "comprehensive knowledge, his trenchant wit and almost miraculous lucidity of his mathematical writings."

Thus died Henri Poincaré who represented "all that was the purest, the best and the most disinterested, in the genius of France." And of his productions one may say, as has been said of the productions of a kindred soul "he who drinks at such fountains can never grow old, for the clear waters which flow therefrom are the true elixir of the human spirit."

On the Existence of the mean Differential Coefficent of a Continuous Function.

By

Ganesh Prasad.

The object of the present note is first to point out a remarkable peculiarity of the well-known non-differentiable functions, and then to show how we can decide whether a continuous function $f(x)$ exists for which the mean differential coefficient, *viz.* $\lim\limits_{h=0} \dfrac{f(x+h)-f(x-h)}{2h}$, is non-existent for *every* value of x.

1. Let $f(x)=\sum\limits_{m=1}^{\infty} a_m u_m(x)$ represent any of the well-known non-differentiable functions including those given by Weierstrass, Darboux, Lerch,* Faber ** and Landsberg.*** Then it is easy to see (1) that if a_m is a minimum or maximum of u_m, so is it also of u_{m+p} whatever integral value p may have; (2) that, h being sufficiently small,

$$u_{m+p}(a_m+h)=u_{m+p}(a_m-h).$$

Hence

$$\frac{f(a_m+h)-f(a_m-h)}{2h}=\sum_{r=1}^{m-1} a_r \frac{u_r(a_m+h)-u_r(a_m-h)}{2h}$$

and therefore the mean differential coefficient is existent for $x=a_m$. It is obvious that $\{a_m\}$ is an everywhere dense aggregate. Thus it is proved that, corresponding to each of the well-known non-differentiable functions, there exists an everywhere dense aggregate of values of x for each of which the mean differential coefficient is existent. As an example of such an aggregate, may be mentioned $\left\{\dfrac{M}{13^m}\right\}$ corresponding to Weierstrass's function $\sum\limits_{1}^{\infty}\dfrac{1}{2^m}\cos(13^m \pi x)$, M being any odd number prime to 13.

* For the functions of Weierstrass, Darboux and Lerch, see Dini-Luroth '*Grundlagen f. e. Theorie d. Functionen e. veranderlichen Grosse*" (1892), pages 223 and 229.

** "Simple example of a continuous and nowhere differentiable function" (In German), *Jahresbericht d. d. Math.-Vereinigung*, Bd. 16. (1907); also *Mathematische Annalen*, Bd. 66. (1909).

*** "On the differentiability of continuous functions" (In German), *Jahresbericht d. d. Math.-Vereinigung*, Bd. 17. (1908).

2. The question naturally arises: Does there exist a continuous function which has no mean differential co-efficient for *any* value of x? Before deciding how this question can be solved, it must be clearly understood that, although the existence of the mean differential co-efficient follows from the existence of the ordinary differential co-efficient, the converse is not true.* Thus it is not inconceivable that the answer to the question may be in the negative. However, the proper procedure for arriving at a decision, as to what the answer should be, is to examine carefully *all* the known non-differentiable functions with a view to determine whether they have the peculiarity mentioned in Art. 1, and if this be the case, to determine whether any modification (in any of the functions under examination) will result in giving a non-differentiable function which has no mean differential co-efficient anywhere. It should be noted that the examination, referred to above, is rendered possible by the fact that almost all the non-differentiable functions, including those recently given by Faber and Landsberg, may be regarded as particular cases of the general non-differentiable function first studied by Dini.†

* As is clear from the case of $x \sin \frac{1}{x}$ which has the mean differential co-efficient for $x=0$ but not the ordinary differential co-efficient

† *Annali di Matematica*, Ser. 2,t.8.; also "*Grundlagen*", ch.10.

Review.

Vectorial Mechanics. By L. Silberstein, Ph.D., (Berlin) (Macmillan & Co. Limited, London, 1913) viii + 197 *pp.* Price 7s. 6d. net.

Prof. Silberstein has done a great service to the Mathematical world by contributing to the diffusion of the elegant, compact, inartificial and in most cases, simpler method of vectors. That the method of vectors has not as yet gained for itself as many votaries as it undoubtedly deserves is due in a large measure to the want of a sufficient number of English text-books on the subject. We must therefore welcome another book specially when, as its novel feature, it is solely devoted to applications to Mathematical Physics.

The size of the book hardly indicates the wide range of subjects it deals with. Indeed, the author has remarkably succeeded in dealing in a very short compass with almost all the classical, fundamental and general conceptions in Natural Philosophy, thus clearly demonstrating the value of the use of vectors. The unique feature of the book which makes it specially valuable as a pocket-map to the Mathematical Physicist is the author's attempt at a rigorous and systematic exposition, step by step, of the General and Special Principles in Mechanics, and of all the principal results in Dynamics of Rigid and of Deformable Bodies and Hydrodynamics.

The first chapter is meant for those uninitiated in Vector Algebra and Analysis. The author has done well in not devoting too much space to them. The notation is that of Heaviside; it has obvious defects which of course would not be admitted by Heaviside and his followers. Heaviside's notation goes on very well so long as one has not to write; the letters S, T, U which Heaviside discards cannot but help the imagination. The scalar product has been defined as A B cos θ with the false non-quaternionic results $i^2=j^2=k^2=+1$, and we are thus led to a non-associative Algebra of appalling complexity and to an artificial inefficient nabla ∇. The author, after giving as much of Vector Algebra as is required for the purposes of the book, passes on to the Differential and Integral properties of Vectors. Here he considers certain Kinematical Problems in connection with differential properties, introduces line and surface integrals and discusses at length the conceptions of circulation, curl, divergence, Hamiltonian and other associated ideas. At this stage, we find inaccuracies here and there. For example, at *p.*33 he speaks of "ψ a scalar, as the *projection* of a certain

vector C upon the normal n, *i.e.* as the normal *Component* of C : ψ=Cn", when he meant to speak of ψ as the scalar product of C and n. Again no attempt has been made at a rigorous proof of Stokes' theorem.

In the second chapter, the author rather hurriedly passes over the three General Principles of Mechanics *viz.* D'Alembert's Principle, Lagrange's Equations and Hamilton's Principle, and in the third chapter, the three special Principles of Vis-Viva, Centre of Gravity and Areas are deduced from D'Alembert. The book as the author says in the preface is intended not only for those well grounded in Dynamical Principles but also those who are acquainted with little more than D'Alembert's Principle. If the book is to be really appreciated by readers of the latter description, a few more examples illustrating the Principles in these as well as in the succeeding chapters, without adding considerably to the volume of the book, seem to be essential.

In the fourth chapter, the author passes on to a neat treatment of Rigid Dynamics, and it is interesting to note how beautifully Euler's Equations of motion, Motion a la Poinsot and motion of Rotation under Gravity have been adapted to vectorial treatment. When closing the algebra of Vectors in the first chapter, the author promised a treatment of symmetrical linear vector operators in the chapter on Rigid Dynamics, but in the latter chapter, the author assumed the fundamental notions of such operators, with the consequence that the reader unacquainted with such operation might find himself at sea.

The fifth chapter begins with a short exposition of the General or non-symmetrical linear vector operator and then follow in order strains, both finite and infinitesimal, surfaces of discontinuity, kinematics and kinetics of Deformable Bodies, and finally stresses. The author's treatment of this chapter and specially of Hadamard's problem of discontinuous displacements presents novel features.

The last viz., the sixth Chapter has come in for an interesting treatment of almost all the general propositions in fluid motion, which is sufficient to convince the reader as to the superiority of Vector methods. The author begins with certain fundamental notions and equations and goes on in succession to Clebsch's transformations and equations of irrotational motion, and finally almost all the properties of vortex motion are proved. The author has stuck to the old equation $v=\nabla\Phi$ although high authorities have replaced it by the new one $v=-\nabla\Phi$.

The book ends with a number of problems and exercises, and a useful list of Cartesian Equivalents for vector formulæ.

A regrettable feature which the reader cannot fail to notice is the large number of inaccuracies in historical references. Tait's name is often missed, and the credit which is due to him is given to Gibbs or Heaviside. For example, at p. 93 the author attributes the discovery of the linear vector operator after Hamilton to Gibbs. This is hardly correct. Again, Gibbs and Heaviside are mentioned in connection with ∇, but there is not a word for Tait.

On the whole, Dr. Silberstein's is an admirable book, and it cannot but create in the Mathematical Physicist some enthusiasm for vector methods.

MANMATHANATH RAY.

$$\xi_1 = S - a^2\frac{S^3}{3!} - 3aa'\frac{S^4}{4!} + (a^4 - 4aa'' - 3a'^2 + a^2\beta^2)\frac{S^5}{5!} +$$

$S^6/6!$

$+10a^3a'$ $-5aa'''$ $-10a'a''$

$+5a^2\beta\beta'$ $+5aa'\beta^2$

$S^7/7!$

$-a^6$ $+20a^3a''$ $+45a^2a'^2$ $-6aa^{iv}$ $-15a'a'''$ $-10a''^2$

$-2a^4\beta^2$ $-a^2\beta^4$ $+8a^2\beta'^2$ $+9a^2\beta\beta''$ $+32aa'\beta\beta'$ $+11aa''\beta^2$ $+5a'^2\beta^2$

$-a^2\beta^2\gamma^2$

$S^8/8!$

$-21a^5a'$ $+35a^3a'''$ $+210a^2a'a''$ $+105aa'^3$ $-7aa^v$ $-21a'a^{iv}$ $-35a''a'''$

$-14a^4\beta\beta'$ $-28a^3a'\beta^2$ $-14a^2\beta^3\beta'$ $+14a^2\beta\beta'''$ $+35a^2\beta'\beta''$ $-7aa'\beta^4$ $+63aa'\beta'^2$ $+70aa'\beta\beta''$ $+84aa''\beta\beta'$ $+21aa'''\beta^2$ $+42a'^2\beta\beta'$ $+21a'a''\beta^2$

$-7a^2\beta^2\gamma\gamma'$ $-7a^2\beta\beta'\gamma^2$ $-7aa'\beta^2\gamma^2$

$S^9/9!$

$+a^8$ $-56a^5a''$ $-210a^4a'^2$ $+56a^3a^{iv}$ $+280a^2a''^2$ $+420a^2a'a'''$ $+840aa'^2a''$ $-8aa^{vi}$ $+105a'$ $-28a'a^v$ $-56a''a^{iv}$ $-35a'''^2$

$+3a^6\beta^2$ $+3a^4\beta^4$ $-34a^4\beta\beta''$ $-32a^4\beta'^2$ $-226a^3a'\beta\beta'$ $-78a^3a''\beta^2$ $-162a^2a'^2\beta^2$ $+a^2\beta^6$ $-34a^2\beta^3\beta''$ $-87a^2\beta^2\beta'^2$ $+20a^2\beta\beta^{iv}$ $+64a^2\beta'\beta'''$ $+45a^2\beta''^2$ $-116aa'\beta^3\beta'$ $+128aa'\beta\beta'''$ $+326aa'\beta'\beta''$ $-22aa''\beta^4$ $+214aa''\beta\beta''$ $+192aa''\beta'^2$ $+186aa'''\beta\beta'$ $+36aa^{iv}\beta^2$ $-7a'^2\beta^4$ $+112a'^2\beta\beta''$ $+105a'^2\beta'^2$ $+210a'a''\beta\beta'$ $+42a'a'''\beta^2$ $+21a''^2\beta^2$

$+2a^4\beta^2\gamma^2$ $+2a^2\beta^4\gamma^2$ $+a^2\beta^2\gamma^4$ $-16a^2\beta^2\gamma\gamma''$ $-15a^2\beta^2\gamma'^2$ $-60a^2\beta\beta'\gamma\gamma'$ $-18a^2\beta\beta''\gamma^2$ $-12a^2\beta'^2\gamma^2$ $-58aa'\beta^2\gamma\gamma'$ $-58aa'\beta\beta'\gamma^2$ $-22aa''\beta^2\gamma^2$ $-7a'^2\beta^2\gamma^2$

$+a^2\beta^2\gamma^2\delta^2$

$$\xi_2 = a\frac{S^2}{2!} + a'\frac{S^3}{3!} + (-a^3 + a'' - a\beta^2)\frac{S^4}{4!} + (-6a^2a' + a''' - 3a\beta\beta' - 3a'\beta^2)\frac{S^5}{5!} +$$

$S^6/6!$

$+a^5$ $-10a^2a''$ $-15aa'^2$ $+a^{iv}$

$+2a^3\beta^2$ $+a\beta^4$ $-3a\beta'^2$ $-4a\beta\beta''$ $-12a'\beta\beta'$ $-6a''\beta^2$

$+a\beta^2\gamma^2$

$S^7/7!$

$+15a^4a'$ $-60aa'a''$ $-15a'^3$ $+a^v$

$+10a^3\beta\beta'$ $+20a^2a'\beta^2$ $+10a\beta^3\beta'$ $-5a\beta\beta'''$ $-10a\beta'\beta''$ $+5a'\beta^4$ $-20a'\beta\beta''$ $-15a'\beta'^2$ $-30a''\beta\beta'$ $-10a'''\beta^2$

$+5a\beta^2\gamma\gamma'$ $+5a\beta\beta'\gamma^2$

$S^8/8!$

$-a^7$ $+35a^4a''$ $+105a^3a'^2$ $-21a^2a^{iv}$ $-105aa'a'''$ $-70aa''^2$ $-105a'^2a''$ $+a^{vi}$

$-3a^5\beta^2$ $-3a^3\beta^4$ $+18a^3\beta'^2$ $+20a^3\beta\beta''$ $+114a^2a'\beta\beta'$ $+50a^2a''\beta^2$ $-a\beta^6$ $+45a\beta^2\beta'^2$ $+78aa'^2\beta^2$ $-6a\beta\beta^{iv}$ $-15a\beta'\beta'''$ $-10a\beta''^2$ $+60a'\beta^3\beta'$ $-30a'\beta\beta'''$ $-60a'\beta'\beta''$ $+15a''\beta^4$ $-45a''\beta'^2$ $-60a''\beta\beta''$ $-60a'''\beta\beta'$ $-15a^{iv}\beta^2$ $-2a^3\beta^2\gamma^2$ $-2a\beta^4\gamma^2$ $+8a\beta^2\gamma'^2$ $-a\beta\ \gamma^4$

$+32a\beta\beta'\gamma\gamma'$ $+11a\beta\beta''\gamma^2$ $+5a\beta'^2\gamma^2$ $+30a'\beta^2\gamma\gamma'$ $+30a'\beta\beta'\gamma^2$ $+15a''\beta^2\gamma^2$

$-a\beta^2\gamma^2\delta^2$

$S^9/9!$

$-28a^6a'$ $+70a^4a'''$ $+560a^3a'a''$ $+420a^2a'^3$ $-28a^2a^v$ $-168aa'a^{iv}$ $-280aa''a'''$

$-280a'a''^2$	$-21a\beta^5\beta'$	$-105a'\beta'\beta'''$	$-14a\beta^4\gamma\gamma'$	$+63a'\beta^2\gamma\gamma''$
$+a^{vii}$	$+35a\beta^3\beta'''$	$-70a'\beta''^2$	$-28a\beta^3\beta'\gamma^2$	$+56a'\beta^2\gamma'^2$
	$+210a\beta^2\beta'\beta''$	$+210a''\beta^3\beta'$	$-14a\beta^2\gamma^3\gamma'$	$+224a'\beta\beta'\gamma\gamma'$
	$+483aa'^2\beta\beta'$	$-105a''\beta\beta'''$	$+14a\beta^2\gamma\gamma'''$	$+77a'\beta\beta''\gamma^2$
$-21a^5\beta\beta'$	$+105a\beta\beta'^3$	$-210a''\beta'\beta''$	$+35a\beta^2\gamma'\gamma''$	$+35a'\beta'^2\gamma^2$
$-63a^4a'\beta^2$	$-7a\beta\beta^{v}$	$+35a'''\beta^4$	$-7a\beta\beta'\gamma^4$	$+105a''\beta^2\gamma\gamma'$
$-42a^3\beta^3\beta'$	$-21a\beta'\beta^{iv}$	$-140a'''\beta\beta''$	$+70a\beta\beta'\gamma\gamma''$	$+105a''\beta\beta'\gamma^2$
$+35a^3\beta\beta'''$	$-35a\beta''\beta'''$	$-105a'''\beta'^2$	$+63a\beta\beta'\gamma'^2$	$+35a'''\beta^2\gamma^2$
$+91a^3\beta'\beta''$	$+441aa'a''\beta^2$	$-105a^{iv}\beta\beta'$	$+84a\beta\beta''\gamma\gamma'$	
$-42a^2a'\beta^4$	$+126a'^3\beta^2$	$-21a^{v}\beta^2$	$+21a\beta\beta'''\gamma^3$	
$+259a^2a'\beta\beta'''$	$-7a'\beta^6$		$+42a\beta'^2\gamma\gamma'$	$-7a\beta^2\gamma^2\delta\delta'$
$+281a^2a'\beta'^2$	$+140a'\beta^3\beta''$	$-14a^3\beta^2\gamma\gamma'$	$+21a\beta'\beta''\gamma^2$	$-7a\beta^2\gamma\gamma'\delta^2$
$+329a^2a''\beta\beta'$	$+315a'\beta^2\beta'^2$	$-14a^3\beta\beta'\gamma^2$	$-14a'\beta^4\gamma^2$	$-7a\beta\beta'\gamma^2\delta^2$
$+105a^2a'''\beta^2$	$-42a'\beta\beta^{iv}$	$-28a^2a'\beta^2\gamma^2$	$-7a'\beta^2\gamma'^4$	$-7a'\beta\gamma^2\delta^2$

$$\xi_3=a\beta\frac{S^3}{3!}+(a\beta'+2a'\beta)\frac{S^4}{4!}+(-a^3\beta-a\beta^3+a\beta''+3a'\beta'+3a''\beta-a\beta\gamma^2)\frac{S^5}{5!}+$$

$S^6/6!$	$-30a'\beta^2\beta'$	$-a^3\beta'''$	$+15a^2a'\beta\gamma^2$	$S^9/9!$
	$+5a'\beta'''$	$-15a^2a'\beta''$	$+10a\beta^3\gamma\gamma'$	
$-a^3\beta'$	$-10a''\beta^3$	$-31a^2a''\beta'$	$+20a\beta^2\beta'\gamma^2$	$-a^7\beta$
$+a\beta'''$	$+10a''\beta''$	$-34a^2a'''\beta$	$+10a\beta\gamma^3\gamma'$	$-3a^5\beta^3$
$-4a'\beta^3$	$+10a'''\beta'$	$+15a\beta^4\beta'$	$-5a\beta\gamma\gamma'''$	$+a^5\beta''$
$+4a'\beta''$	$+5a^{iv}\beta$	$-15a\beta^2\beta'''$	$-10a\beta\gamma'\gamma''$	$+25a^4a'\beta'$
$+6a''\beta'$		$-57aa'^2\beta'$	$+5a\beta'\gamma^4$	$+55a^4a''\beta$
$+4a'''\beta$	$+a^3\beta\gamma^2$	$-164aa'a''\beta$	$-20a\beta'\gamma\gamma''$	$-3a^3\beta^5$
$-9a^2a'\beta$	$+2a\beta^3\gamma^2$	$-60a\beta\beta'\beta''$	$-15a\beta'\gamma'^2$	$+185a^3a'^2\beta$
$+6a\beta^2\beta'$	$+a\beta\gamma^4$	$-15a\beta'^3$	$-30a\beta''\gamma\gamma'$	$+36a^3\beta^2\beta''$
	$-4a\beta\gamma\gamma''$	$+a\beta^{v}$	$-10a\beta'''\gamma^2$	$+50a^3\beta\beta'^2$
$-3a\beta'\gamma^2$	$-3a\beta\gamma'^2$	$-48a'^3\beta$	$+6a'\beta\gamma^4$	$-a^3\beta^{iv}$
$-4a'\beta\gamma^2$	$-12a\beta'\gamma\gamma'$	$+6a'\beta^5$	$+12a'\beta^3\gamma_2$	$+240a^2a'\beta^2\beta'$
$-3a\beta\gamma\gamma'$	$-6a\beta''\gamma^2$	$-60a'\beta^2\beta''$	$-24a'\beta\gamma\gamma''$	$-18a^2a'\beta''$
$S^7/7!$	$-15a'\beta\gamma\gamma'$	$-90a'\beta\beta'^2$	$-18a'\beta\gamma'^2$	$+76a^2a''\beta^3$
$+a^5\beta$	$-15a'\beta'\gamma^2$	$+6a'\beta^{iv}$	$-72a'\beta'\gamma\gamma'$	$-46a^2a''\beta''$
$+2a^3\beta^3$	$-10a''\beta\gamma^2$	$-90a''\beta^2\beta'$	$-36a'\beta''\gamma^2$	$-65a^2a'''\beta'$
$-a^3\beta''$		$+15a''\beta'''$	$-45a''\beta\gamma\gamma'$	$-55a^2a^{iv}\beta$
$-12a^2a'\beta'$	$+a\beta\gamma^2\delta^2$	$-20a'''\beta^3$	$-45a''\beta'\gamma^2$	$+130aa'^2\beta^3$
$+a\beta^5$		$+20a'''\beta''$	$-20a'''\beta\gamma^2$	$-87aa'^2\beta''$
$-19a^2a''\beta$	$S^8/8!$	$+15a^{iv}\beta'$		$-340aa'a''\beta'$
$-10a\beta^2\beta''$	$+a^5\beta'$	$+6a^{v}\beta$	$+5a\beta\gamma^2\delta\delta'$	$-337aa'a'''\beta$
$-15a\beta\beta'^2$	$+20a^4a'\beta$		$+5a\beta\gamma\gamma'\delta^2$	$-a\beta^7$
$-33aa'^2\beta''$	$+16a^3\beta^2\beta'$	$+3a^3\beta\gamma\gamma'$	$+5a\beta'\gamma^2\delta^2$	$+35a\beta^4\beta''$
$+a\beta^{iv}$	$+26a^3a'\beta^3$	$+3a^3\beta'\gamma^2$	$+6a'\beta\gamma^2\delta^2$	$+105a\beta^3\beta'^2$

$-21\alpha\beta^2\beta^{iv}$	$-35\alpha^{iv}\beta^3$	$+18\alpha\beta^3\gamma'^2$	$+70\alpha'\beta^3\gamma\gamma'$	
$-105\alpha\beta\beta'\beta'''$	$+35\alpha^{iv}\beta''$	$+114\alpha\beta^2\beta'\gamma\gamma'$	$+140\alpha'\beta^2\beta'\gamma^3$	$-\alpha^3\beta\gamma^2\delta^2$
$-70\alpha\beta\beta''^2$	$+21\alpha^{v}\beta'$	$+50\alpha\beta^2\beta''\gamma^2$	$+70\alpha'\beta\gamma^3\gamma'$	$-2\alpha\beta^3\gamma^2\delta^2$
$-105\alpha\beta'^2\beta''$	$+7\alpha^{vi}\beta$	$-\alpha\beta\gamma^6$	$-35\alpha'\beta\gamma\gamma'''$	$-2\alpha\beta\gamma^4\delta^2$
$-234\alpha\alpha''^2\beta$		$+20\alpha\beta\gamma^3\gamma''$	$-70\alpha'\beta\gamma'\gamma''$	$-\alpha\beta\gamma^2\delta^4$
$+\alpha\beta^{vi}$		$+45\alpha\beta\gamma^2\gamma'^2$	$+35\alpha'\beta'\gamma^4$	$+9\alpha\beta\gamma^2\delta\delta''$
$-105\alpha'^3\beta'$	$-\alpha^5\beta\gamma^2$	$-6\alpha\beta\gamma\gamma^{iv}$	$-140\alpha'\beta'\gamma\gamma''$	$+8\alpha\beta\gamma^2\delta'^2$
$-413\alpha'^2\alpha''\beta$	$-4\alpha^3\beta^3\gamma^2$	$+78\alpha\beta\beta'^2\gamma^2$	$-105\alpha'\beta'\gamma'^2$	$+32\alpha\beta\gamma\gamma'\delta\delta'$
$+105\alpha'\beta^4\beta'$	$-\alpha^3\beta\gamma^4$	$-15\alpha\beta\gamma'\gamma'''$	$-210\alpha'\beta''\gamma\gamma''$	$+11\alpha\beta\gamma\gamma''\delta^2$
$-105\alpha'\beta^2\beta'''$	$+4\alpha^3\beta\gamma\gamma''$	$-10\alpha\beta\gamma''^2$	$-70\alpha'\beta'''\gamma^2$	$+5\alpha\beta\gamma'^2\delta^2$
$-420\alpha'\beta\beta'\beta''$	$+3\alpha^3\beta\gamma'^2$	$+87\alpha\alpha'^2\beta\gamma^3$	$+42\alpha''\beta^3\gamma^2$	$+30\alpha\beta'\gamma^2\delta\delta'$
$-105\alpha'\beta'^3$	$+12\alpha^3\beta'\gamma\gamma'$	$+60\alpha\beta'\gamma^3\gamma'$	$+21\alpha''\beta\gamma^4$	$+30\alpha\beta'\gamma\gamma'\delta^2$
$+7\alpha'\beta^{v}$	$+6\alpha^3\beta''\gamma^2$	$-30\alpha\beta'\gamma\gamma'''$	$-84\alpha''\beta\gamma\gamma''$	$+15\alpha\beta''\gamma^2\delta^2$
$+21\alpha''\beta^5$	$+54\alpha^2\alpha'\beta\gamma\gamma'$	$-60\alpha\beta'\gamma'\gamma''$	$-63\alpha''\beta\gamma'^2$	$+35\alpha'\beta\gamma^2\delta\delta'$
$-210\alpha''\beta^2\beta''$	$+54\alpha^2\alpha'\beta'\gamma^2$	$+15\alpha\beta''\gamma^4$	$-252\alpha''\beta'\gamma\gamma'$	$+35\alpha'\beta\gamma\gamma'\delta$
$-315\alpha''\beta\beta'^2$	$+46\alpha^2\alpha''\beta\gamma^2$	$-60\alpha\beta''\gamma\gamma''$	$-126\alpha''\beta''\gamma^2$	$+35\alpha'\beta'\gamma^2\delta^2$
$+21\alpha''\beta^{iv}$	$-3\alpha\beta^5\gamma^2$	$-45\alpha\beta''\gamma'^2$	$-105\alpha'''\beta\gamma\gamma'$	$+21\alpha''\beta\gamma^2\delta^2$
$-210\alpha'''\beta^2\beta'$	$-3\alpha\beta^3\gamma^4$	$-60\alpha\beta'''\gamma\gamma'$	$-105\alpha'''\beta'\gamma^2$	
$+35\alpha'''\beta'''$	$+20\alpha\beta^3\gamma\gamma''$	$-15\alpha\beta^{iv}\gamma^2$	$-35\alpha^{iv}\beta\gamma^3$	$-\alpha\beta\gamma^2\delta^2\epsilon^2$

$$\xi_4=\alpha\beta\gamma\cdot\frac{S^4}{4!}+(\alpha\beta\gamma'+2\alpha\beta'\gamma+3\alpha'\beta\gamma)\frac{S^5}{5!}+$$

$S^6/6!$	$-4\alpha\beta'\gamma$	$-3\alpha^3\beta'\gamma'$	$-6\alpha'\beta^3\gamma'$	$-4\alpha\beta\gamma\delta\delta''$
	$+4\alpha\beta'\gamma''$	$-3\alpha^3\beta''\gamma$	$-54\alpha'\beta^2\beta'\gamma$	$-12\alpha\beta\gamma'\delta\delta'$
$-\alpha^8\beta\gamma$	$+6\alpha\beta''\gamma'$	$-15\alpha^2\alpha'\beta\gamma'$	$-36\alpha'\beta\gamma^2\gamma'$	$-6\alpha\beta\gamma''\delta^2$
$-\alpha\beta^3\gamma$	$+4\alpha\beta'''\gamma$	$-30\alpha^2\alpha'\beta'\gamma$	$+6\alpha'\beta\gamma'''$	$-15\alpha\beta'\gamma\delta\delta'$
$-\alpha\beta\gamma^3$	$-5\alpha'\beta^3\gamma$	$-31\alpha^2\alpha''\beta\gamma$	$-24\alpha'\beta'\gamma^3$	$-15\alpha\beta'\gamma'\delta^2$
$+\alpha\beta\gamma''$	$-5\alpha'\beta\gamma^3$	$+\alpha\beta^5\gamma$	$+24\alpha'\beta'\gamma''$	$-10\alpha\beta''\gamma\delta^2$
$+3\alpha\beta'\gamma'$	$+5\alpha'\beta\gamma''$	$+2\alpha\beta^3\gamma^3$	$+36\alpha'\beta''\gamma'$	$-18\alpha'\beta\gamma\delta\delta'$
$+3\alpha\beta''\gamma$	$+15\alpha'\beta'\gamma'$	$-\alpha\beta^3\gamma''$	$+24\alpha'\beta'''\gamma$	$-18\alpha'\beta\gamma'\delta^2$
$+4\alpha'\beta\gamma'$	$+15\alpha'\beta''\gamma$	$-12\alpha\beta^2\beta'\gamma'$	$-15\alpha''\beta^3\gamma$	$-24\alpha'\beta'\gamma\delta^2$
$+8\alpha'\beta'\gamma$	$+10\alpha''\beta\gamma'$	$-19\alpha\beta^2\beta''\gamma$	$-15\alpha''\beta\gamma^3$	$-15\alpha''\beta\gamma\delta^2$
$+6\alpha''\beta\gamma$	$+20\alpha''\beta'\gamma$	$+\alpha\beta\gamma^5$	$+15\alpha''\beta\gamma''$	
	$+10\alpha'''\beta\gamma$	$-57\alpha\alpha'^2\beta\gamma$	$+45\alpha''\beta'\gamma'$	$+\alpha\beta\gamma\delta^2\epsilon^2$
$-\alpha\beta\gamma\delta^2$	$-3\alpha\beta\gamma'\delta^2$	$-33\alpha\beta\beta'^2\gamma$	$+45\alpha''\beta''\gamma$	$S^9/9!$
$S^7/7!$	$-4\alpha\beta'\gamma\delta^2$	$-10\alpha\beta\gamma^2\gamma''$	$+20\alpha'''\beta\gamma'$	
	$-3\alpha\beta\gamma\delta\delta'$	$-15\alpha\beta\gamma\gamma'^2$	$+40\alpha'''\beta'\gamma$	$+\alpha^5\beta\gamma'$
$-\alpha^3\beta\gamma'$	$-5\alpha'\beta\gamma\delta^2$	$+\alpha\beta\gamma^{iv}$	$+15\alpha^{iv}\beta\gamma$	$+2\alpha^5\beta'\gamma$
$-2\alpha^3\beta'\gamma$	$S^8/8!$	$-30\alpha\beta'\gamma^2\gamma'$		$+25\alpha^4\alpha'\beta\gamma$
$-12\alpha^2\alpha'\beta\gamma$		$+5\alpha\beta'\gamma'''$	$+\alpha^3\beta\gamma\delta^2$	$+2\alpha^3\beta^3\gamma'$
$-\alpha\beta^3\gamma'$	$+\alpha^5\beta\gamma$	$-10\alpha\beta''\gamma^3$	$+\alpha\beta^3\gamma\delta^2$	$+22\alpha^3\beta^2\beta\gamma$
$-9\alpha\beta^2\beta'\gamma$	$+2\alpha^3\beta^3\gamma$	$+10\alpha\beta''\gamma''$	$+2\alpha\beta\gamma^3\delta^2$	$+6\alpha^3\beta\gamma^2\gamma'$
$-6\alpha\beta\gamma^2\gamma'$	$+\alpha^3\beta\gamma^3$	$+10\alpha\beta'''\gamma'$	$+\alpha\beta\gamma\delta^4$	$-\alpha^3\beta\gamma'''$
$+\alpha\beta\gamma'''$	$-\alpha^3\beta\gamma''$	$+5\alpha\beta^{iv}\gamma$	$-3\alpha\beta\gamma\delta'^2$	$+4\alpha^3\beta'\gamma^3$

$-4\alpha^3\beta'\gamma''$	$+15\alpha\beta\gamma^4\gamma'$	$-70\alpha'\beta\gamma^2\gamma''$	$+21\alpha^{v}\beta\gamma$	$-36\alpha\beta'\gamma''\delta^2$
$-6\alpha^3\beta''\gamma'$	$-15\alpha\beta\gamma^2\gamma'''$	$-105\alpha'\beta\gamma\gamma'^2$		$-45\alpha\beta''\gamma\delta\delta'$
$-4\alpha^3\beta'''\gamma$	$-60\alpha\beta\gamma\gamma'\gamma''$	$+7\alpha'\beta\gamma^{iv}$	$+3\alpha^3\beta\gamma\delta\delta'$	$-45\alpha\beta''\gamma'\delta^2$
$+32\alpha^2\alpha'\beta^3\gamma$	$-15\alpha\beta\gamma'^3$	$-210\alpha'\beta'\gamma^2\gamma'$	$+3\alpha^3\beta\gamma'\delta^2$	$-20\alpha\beta'''\gamma\delta^2$
$+18\alpha^2\alpha'\beta\gamma^3$	$+\alpha\beta\gamma^{v}$	$+35\alpha'\beta'\gamma'''$	$+4\alpha^3\beta'\gamma\delta^2$	$+7\alpha'\beta^3\gamma\delta^2$
$-18\alpha^2\alpha'\beta\gamma''$	$-48\alpha\beta'^3\gamma$	$-60\alpha'\beta''\gamma^3$	$+18\alpha^2\alpha'\beta\gamma\delta^2$	$+14\alpha'\beta\gamma^3\delta^2$
$-54\alpha^2\alpha'\beta'\gamma'$	$+6\alpha\beta'\gamma^5$	$+60\alpha'\beta''\gamma''$	$+3\alpha\beta^3\gamma\delta\delta'$	$+7\alpha'\beta\gamma\delta^4$
$-54\alpha^2\alpha'\beta''\gamma$	$-60\alpha\beta'\gamma^2\gamma''$	$+60\alpha'\beta'''\gamma'$	$+3\alpha\beta^3\gamma'\delta^2$	$-21\alpha'\beta\gamma\delta'^2$
$-46\alpha^2\alpha''\beta\gamma'$	$-90\alpha\beta'\gamma\gamma'^2$	$+35\alpha'\beta^{iv}\gamma$	$+15\alpha\beta^2\beta'\gamma\delta^2$	$-28\alpha'\beta\gamma\delta\delta''$
$-92\alpha^2\alpha''\beta'\gamma$	$+6\alpha\beta'\gamma^{iv}$	$-21\alpha''\beta^3\gamma'$	$+20\alpha\beta\gamma^3\delta\delta'$	$-84\alpha'\beta\gamma'\delta\delta'$
$-65\alpha^2\alpha'''\beta\gamma$	$-90\alpha\beta''\gamma^2\gamma'$	$-174\alpha''\beta^2\beta'\gamma$	$+20\alpha\beta\gamma^2\gamma'\delta^2$	$-42\alpha'\beta\gamma''\delta^2$
$-87\alpha\alpha'^2\beta\gamma'$	$+15\alpha\beta''\gamma''$	$-126\alpha''\beta\gamma^2\gamma'$	$+10\alpha\beta\gamma\delta^3\delta'$	$-105\alpha'\beta'\gamma\delta\delta'$
$-174\alpha\alpha'^2\beta'\gamma$	$-20\alpha\beta'''\gamma^3$	$+21\alpha''\beta\gamma'''$	$-10\alpha\beta\gamma\delta'\delta''$	$-105\alpha'\beta'\gamma'\delta^2$
$-340\alpha\alpha'\alpha''\beta\gamma$	$+20\alpha\beta'''\gamma''$	$-84\alpha''\beta'\gamma^3$	$-5\alpha\beta\gamma\delta\delta'''$	$-70\alpha'\beta''\gamma\delta^2$
$+\alpha\beta^5\gamma'$	$+15\alpha\beta^{iv}\gamma'$	$+84\alpha''\beta'\gamma''$	$+5\alpha\beta\gamma'\delta^4$	$-63\alpha''\beta\gamma\delta\delta'$
$+20\alpha\beta^4\beta'\gamma$	$+6\alpha\beta^{v}\gamma$	$+126\alpha''\beta''\gamma'$	$-15\alpha\beta\gamma'\delta'^2$	$-63\alpha''\beta\gamma'\delta^2$
$+16\alpha\beta^3\gamma^2\gamma'$	$-105\alpha'^3\beta\gamma$	$+84\alpha''\beta'''\gamma$	$-20\alpha\beta\gamma'\delta\delta''$	$-84\alpha''\beta'\gamma\delta^2$
$-\alpha\beta^3\gamma'''$	$+7\alpha'\beta^5\gamma$	$-21\alpha'''\beta^3\gamma$	$-30\alpha\beta\gamma''\delta\delta'$	$-35\alpha'''\beta\gamma\delta^2$
$+26\alpha\beta^2\beta'\gamma^3$	$+14\alpha'\beta^3\gamma^3$	$-35\alpha'''\beta\gamma^3$	$-10\alpha\beta\gamma'''\delta^2$	
$-15\alpha\beta^2\beta'\gamma''$	$-7\alpha'\beta^3\gamma''$	$+35\alpha'''\beta\gamma''$	$+12\alpha\beta'\gamma^3\delta^2$	$+5\alpha\beta\gamma\delta^2\epsilon\epsilon'$
$-31\alpha\beta^2\beta''\gamma'$	$-84\alpha'\beta^2\beta'\gamma'$	$+105\alpha'''\beta'\gamma'$	$+6\alpha\beta'\gamma\delta^4$	$+5\alpha\beta\gamma\delta\delta'\epsilon^2$
$-35\alpha\beta^2\beta'''\gamma$	$-133\alpha'\beta^2\beta''\gamma$	$+105\alpha'''\beta''\gamma$	$-18\alpha\beta'\gamma\delta'^2$	$+5\alpha\beta\gamma'\delta^2\epsilon^2$
$-57\alpha\beta\beta'^2\gamma'$	$-231\alpha'\beta\beta'^2\gamma$	$+35\alpha^{iv}\beta\gamma'$	$-24\alpha\beta'\gamma\delta\delta''$	$+6\alpha\beta'\gamma\delta^2\epsilon^2$
$-164\alpha\beta\beta'\beta''\gamma$	$+7\alpha'\beta\gamma^5$	$+70\alpha^{iv}\beta'\gamma$	$-72\alpha\beta'\gamma'\delta\delta'$	$+7\alpha'\beta\gamma\delta^2\epsilon^2$

$$\xi_5 = \alpha\beta\gamma\delta\frac{S^5}{5!} + (\alpha\beta\gamma\delta' + 2\alpha\beta\gamma'\delta + 3\alpha\beta'\gamma\delta + 4\alpha'\beta\gamma\delta)\frac{S^6}{6!} +$$

$S^7/7!$	$-\alpha\beta\gamma\delta\epsilon^2$	$+4\alpha\beta\gamma'''\delta$	$+36\alpha'\beta''\gamma\delta$	$+\alpha\beta^3\gamma\delta$
	$S^8/8!$	$-5\alpha\beta'\gamma^3\delta$	$+15\alpha''\beta\gamma\delta'$	$+2\alpha\beta^3\gamma^3\delta$
$-\alpha^3\beta\gamma\delta$		$-5\alpha\beta'\gamma\delta^3$	$+30\alpha''\beta\gamma'\delta$	$+\alpha\beta^3\gamma\delta^3$
$-\alpha\beta^3\gamma\delta$	$-\alpha^3\beta\gamma\delta'$	$+5\alpha\beta'\gamma\delta''$	$+45\alpha''\beta'\gamma\delta$	$+\alpha\beta\gamma^5\delta$
$-\alpha\beta\gamma^3\delta$	$-2\alpha^3\beta\gamma'\delta$	$+15\alpha\beta'\gamma'\delta'$	$+20\alpha'''\beta\gamma\delta$	$+2\alpha\beta\gamma^3\delta^3$
$-\alpha\beta\gamma\delta^3$	$-3\alpha^3\beta'\gamma\delta$	$+15\alpha\beta'\gamma''\delta$		$+\alpha\beta\gamma\delta^5$
$+\alpha\beta\gamma\delta''$	$-15\alpha^2\alpha'\beta\gamma\delta$	$+10\alpha\beta''\gamma\delta'$	$-3\alpha\beta\gamma\delta'\epsilon^2$	$-\alpha^3\beta\gamma\delta''$
$+3\alpha\beta\gamma''\delta$	$-\alpha\beta^3\gamma\delta'$	$+20\alpha\beta''\gamma'\delta$	$-3\alpha\beta\gamma\delta\epsilon\epsilon'$	$-3\alpha^3\beta\gamma'\delta'$
$+3\alpha\beta\gamma'\delta'$	$-2\alpha\beta^3\gamma'\delta$	$+10\alpha\beta'''\gamma\delta$	$-4\alpha\beta\gamma'\delta\epsilon^2$	$-3\alpha^3\beta\gamma''\delta$
$+4\alpha\beta'\gamma\delta'$	$-12\alpha\beta^2\beta'\gamma\delta$	$-6\alpha'\beta^3\gamma\delta$	$-5\alpha\beta'\gamma\delta\epsilon^2$	$-4\alpha^3\beta'\gamma\delta'$
$+8\alpha\beta'\gamma'\delta$	$-\alpha\beta\gamma^3\delta'$	$-6\alpha'\beta\gamma^3\delta$	$-6\alpha'\beta\gamma\delta\epsilon^2$	$-8\alpha^3\beta'\gamma'\delta$
$+6\alpha\beta''\gamma\delta$	$-9\alpha\beta\gamma^2\gamma'\delta$	$-6\alpha'\beta\gamma\delta^3$	$S^9/9!$	$-6\alpha^3\beta''\gamma\delta$
$+5\alpha'\beta\gamma\delta'$	$-6\alpha\beta\gamma\delta^2\delta'$	$+6\alpha'\beta\gamma\delta''$		$-18\alpha^2\alpha'\beta\gamma\delta'$
$+10\alpha'\beta\gamma'\delta$	$+\alpha\beta\gamma\delta'''$	$+18\alpha'\beta\gamma'\delta'$	$+\alpha^5\beta\gamma\delta$	$-36\alpha^2\alpha'\beta\gamma'\delta$
$+15\alpha'\beta'\gamma\delta$	$-4\alpha\beta\gamma'\delta^3$	$+18\alpha'\beta\gamma''\delta$	$+2\alpha^3\beta^3\gamma\delta$	$-54\alpha^2\alpha'\beta'\gamma\delta$
$+10\alpha''\beta\gamma\delta$	$+4\alpha\beta\gamma'\delta''$	$+24\alpha'\beta'\gamma\delta'$	$+\alpha^3\beta\gamma^3\delta$	$-46\alpha^2\alpha''\beta\gamma\delta$
	$+6\alpha\beta\gamma''\delta'$	$+48\alpha'\beta'\gamma'\delta$	$+\alpha^3\beta\gamma\delta^3$	$-87\alpha\alpha'^2\beta\gamma\delta$

$-a\beta^3\gamma\delta''$	$+5a\beta\gamma^{iv}\delta$	$-84a'\beta^2\beta'\gamma\delta$	$-21a''\beta\gamma\delta^3$	$-12a\beta\gamma\delta'\epsilon\epsilon'$
$-3a\beta^2\gamma'\delta'$	$-6a\beta'\gamma^3\delta'$	$-7a'\beta\gamma^3\delta'$	$+21a''\beta\gamma\delta''$	$-6a\beta\gamma\delta''\epsilon^2$
$-3a\beta^3\gamma''\delta$	$-54a\beta'\gamma^2\gamma'\delta$	$-63a'\beta\gamma^2\gamma'\delta$	$+63a''\beta\gamma'\delta'$	$-15a\beta\gamma'\delta\epsilon\epsilon'$
$-15a\beta^2\beta'\gamma\delta'$	$-36a\beta'\gamma\delta^2\delta'$	$-42a'\beta\gamma\delta^2\delta'$	$+63a''\beta\gamma''\delta$	$-15a\beta\gamma'\delta'\epsilon^2$
$-30a\beta^2\beta'\gamma'\delta$	$+6a\beta'\gamma\delta'''$	$+7a'\beta\gamma\delta'''$	$+84a''\beta'\gamma\delta'$	$-10a\beta\gamma''\delta\epsilon^2$
$-31a\beta^2\beta''\gamma\delta$	$-24a\beta'\gamma'\delta^3$	$-28a'\beta\gamma'\delta^3$	$+168a''\beta'\gamma'\delta$	$-18a\beta'\gamma\delta\epsilon\epsilon'$
$-57a\beta\beta'^2\gamma\delta$	$+24a\beta'\gamma'\delta''$	$+28a'\beta\gamma'\delta''$	$+126a''\beta''\gamma\delta$	$-18a\beta'\gamma\delta'\epsilon$
$-a\beta\gamma^3\delta''$	$+36a\beta'\gamma''\delta'$	$+42a'\beta\gamma''\delta'$	$+35a'''\beta\gamma\delta'$	$-24a\beta'\gamma'\delta\epsilon^2$
$-12a\beta\gamma^2\gamma'\delta'$	$+24a\beta'\gamma'''\delta$	$+28a'\beta\gamma'''\delta$	$+70a'''\beta\gamma'\delta$	$-15a\beta''\gamma\delta\epsilon^2$
$-19a\beta\gamma^2\gamma''\delta$	$-15a\beta''\gamma^3\delta$	$-35a'\beta'\gamma^3\delta$	$+105a'''\beta'\gamma\delta$	$-21a'\beta\gamma\delta\epsilon\epsilon'$
$-33a\beta\gamma\gamma'^2\delta$	$-15a\beta''\gamma\delta^3$	$-35a'\beta'\gamma\delta^3$	$+35a^{iv}\beta\gamma\delta$	$-21a'\beta\gamma\delta'\epsilon^2$
$-10a\beta\gamma\delta^2\delta''$	$+15a\beta''\gamma\delta''$	$+35a'\beta'\gamma\delta''$		$-28a'\beta\gamma'\delta\epsilon^2$
$-15a\beta\gamma\delta\delta'^2$	$+45a\beta''\gamma'\delta'$	$+105a'\beta'\gamma'\delta'$	$+a^3\beta\gamma\delta\epsilon^2$	$-35a'\beta'\gamma\delta\epsilon^2$
$+a\beta\gamma\delta^{iv}$	$+45a\beta''\gamma''\delta$	$+105a'\beta'\gamma''\delta$	$+a\beta^3\gamma\delta\epsilon^2$	$-21a''\beta\gamma\delta\epsilon^2$
$-30a\beta\gamma'\delta^2\delta'$	$+20a\beta'''\gamma\delta'$	$+70a'\beta''\gamma\delta'$	$+a\beta\gamma^3\delta\epsilon^2$	
$+5a\beta\gamma'\delta'''$	$+40a\beta'''\gamma'\delta$	$+140a'\beta''\gamma'\delta$	$+2a\beta\gamma\delta^3\epsilon^2$	
$-10a\beta\gamma''\delta^3$	$+15a\beta^{iv}\gamma\delta$	$+70a'\beta'''\gamma\delta$	$+a\beta\gamma\delta\epsilon^4$	$+a\beta\gamma\delta\epsilon^2\theta^2$
$+10a\beta\gamma''\delta''$	$-7a'\beta^3\gamma\delta'$	$-21a''\beta^3\gamma\delta$	$-3a\beta\gamma\delta\epsilon'^2$	
$+10a\beta\gamma'''\delta'$	$-14a'\beta^3\gamma'\delta$	$-21a''\beta\gamma^3\delta$	$-4a\beta\gamma\delta\epsilon\epsilon''$	

$$\xi_6 = a\beta\gamma\delta\epsilon\frac{S^6}{6!} +$$

$S^7/7!$	$+10a\beta'\gamma\delta'\epsilon$	$-15a\beta^2\beta'\gamma\delta\epsilon$	$-6a\beta'\gamma^3\delta\epsilon$	$+56a'\beta\gamma'\delta'\epsilon$
$+a\beta\gamma\delta\epsilon'$	$+15a\beta'\gamma'\delta\epsilon$	$-a\beta\gamma^3\delta\epsilon'$	$-6a\beta'\gamma\delta^3\epsilon$	$+21a'\beta\gamma\delta''\epsilon$
$+2a\beta\gamma\delta'\epsilon$	$+10a\beta''\gamma\delta\epsilon$	$-2a\beta\gamma^3\delta'\epsilon$	$-6a\beta'\gamma\delta\epsilon^3$	$+42a'\beta\gamma''\delta\epsilon$
$+3a\beta\gamma'\delta\epsilon$	$+6a'\beta\gamma\delta\epsilon'$	$-12a\beta\gamma^2\gamma'\delta\epsilon$	$+6a\beta'\gamma\delta\epsilon''$	$+36a'\beta'\gamma\delta\epsilon'$
$+4a\beta'\gamma\delta\epsilon$	$+12a'\beta\gamma\delta'\epsilon$	$-a\beta\gamma\delta^3\epsilon'$	$+18a\beta'\gamma\delta'\epsilon'$	$+70a'\beta'\gamma\delta'\epsilon$
$+5a'\beta\gamma\delta\epsilon$	$+18a'\beta\gamma'\delta\epsilon$	$-9a\beta\gamma\delta^2\delta'\epsilon$	$+48a\beta'\gamma'\delta'\epsilon$	$+105a'\beta'\gamma'\delta\epsilon$
$S^8/8!$	$+24a'\beta'\gamma\delta\epsilon$	$-6a\beta\gamma\delta\epsilon^2\epsilon'$	$+24a\beta'\gamma'\delta\epsilon'$	$+70a'\beta''\gamma\delta\epsilon$
	$+15a''\beta\gamma\delta\epsilon$	$+a\beta\gamma\delta\epsilon'''$	$+18a\beta'\gamma\delta''\epsilon$	$+21a''\beta\gamma\delta\epsilon'$
$-a^3\beta\gamma\delta\epsilon$		$+4a\beta\gamma\delta'\epsilon''$	$+36a\beta'\gamma''\delta\epsilon$	$+42a''\beta\gamma\delta'\epsilon$
$+a\beta^3\gamma\delta\epsilon$	$-a\beta\gamma\delta\epsilon\theta^2$	$+6a\beta\gamma\delta''\epsilon'$	$+15a\beta''\gamma\delta\epsilon'$	$+63a''\beta\gamma'\delta\epsilon$
$-a\beta\gamma^3\delta\epsilon$	$S^9/9!$	$+4a\beta\gamma\delta'''\epsilon$	$+30a\beta''\gamma\delta'\epsilon$	$+84a''\beta'\gamma\delta\epsilon$
$-a\beta\gamma\delta^3\epsilon$		$-4a\beta\gamma\delta'\epsilon^3$	$+45a\beta''\gamma'\delta\epsilon$	$+35a'''\beta\gamma\delta\epsilon$
$-a\beta\gamma\delta\epsilon^3$	$-a^3\beta\gamma\delta\epsilon'$	$-5a\beta\gamma'\delta^3\epsilon$	$+20a\beta'''\gamma\delta\epsilon$	
$+a\beta\gamma\delta\epsilon''$	$-2a^3\beta\gamma\delta'\epsilon$	$-5a\beta\gamma'\delta\epsilon^3$	$-7a'\beta^3\gamma\delta\epsilon$	$-3a\beta\gamma\delta\epsilon'\theta^2$
$+3a\beta\gamma\delta'\epsilon'$	$-3a^3\beta\gamma'\delta\epsilon$	$+5a\beta\gamma'\delta\epsilon''$	$-7a'\beta\gamma^3\delta\epsilon$	$-3a\beta\gamma\delta\epsilon\theta\theta'$
$+3a\beta\gamma\delta''\epsilon$	$-4a^3\beta'\gamma\delta\epsilon$	$+15a\beta\gamma'\delta'\epsilon'$	$-7a'\beta\gamma\delta^3\epsilon$	$-4a\beta\gamma\delta'\epsilon\theta^2$
$+4a\beta\gamma'\delta\epsilon'$	$-18a^2a'\beta\gamma\delta\epsilon$	$+15a\beta\gamma'\delta''\epsilon$	$-7a'\beta\gamma\delta\epsilon^3$	$-5a\beta\gamma'\delta\epsilon\theta^2$
$+8a\beta\gamma'\delta'\epsilon$	$-a\beta^3\gamma\delta\epsilon'$	$+10a\beta\gamma''\delta\epsilon'$	$+7a'\beta\gamma\delta\epsilon''$	$-6a\beta'\gamma\delta\epsilon\theta^2$
$+6a\beta\gamma''\delta\epsilon$	$-2a\beta^3\gamma\delta'\epsilon$	$+20a\beta\gamma''\delta'\epsilon$	$+21a'\beta\gamma\delta'\epsilon'$	$-7a'\beta\gamma\delta\epsilon\theta^2$
$+5a\beta''\gamma\delta\epsilon'$	$-3a\beta^3\gamma'\delta\epsilon$	$+10a\beta\gamma'''\delta\epsilon$	$+28a'\beta\gamma'\delta\epsilon'$	

$$\xi_7 = \alpha\beta\gamma\delta\epsilon\theta.\frac{S^7}{7!}+$$

$S^8/8!$	$S^9/9!$	$+\alpha\beta\gamma\delta\epsilon\theta''$	$+15\alpha\beta\gamma'\delta'\epsilon\theta$	$+14\alpha'\beta\gamma\delta\epsilon'\theta$
		$+3\alpha\beta\gamma\delta\epsilon'\theta'$	$+10\alpha\beta\gamma''\delta\epsilon\theta$	$+21\alpha'\beta\gamma\delta'\epsilon\theta$
$+\alpha\beta\gamma\delta\epsilon\theta'$	$-\alpha^3\beta\gamma\delta\epsilon\theta$	$+3\alpha\beta\gamma\delta\epsilon''\theta$	$+6\alpha\beta'\gamma\delta\epsilon\theta'$	$+28\alpha'\beta\gamma'\delta\epsilon\theta$
$+2\alpha\beta\gamma\delta\epsilon'\theta$	$-\alpha\beta^3\gamma\delta\epsilon\theta$	$+4\alpha\beta\gamma\delta'\epsilon\theta'$	$+12\alpha\beta'\gamma\delta\epsilon'\theta$	$+35\alpha'\beta'\gamma\delta\epsilon\theta$
$+3\alpha\beta\gamma\delta'\epsilon\theta$	$-\alpha\beta\gamma^3\delta\epsilon\theta$	$+8\alpha\beta\gamma\delta'\epsilon'\theta$	$+18\alpha\beta'\gamma\delta'\epsilon\theta$	$+21\alpha''\beta\gamma\delta\epsilon\theta$
$+4\alpha\beta\gamma'\delta\epsilon\theta$	$-\alpha\beta\gamma\delta^3\epsilon\theta$	$+6\alpha\beta\gamma\delta''\epsilon\theta$	$+24\alpha\beta'\gamma'\delta\epsilon\theta$	
$+5\alpha\beta'\gamma\delta\epsilon\theta$	$-\alpha\beta\gamma\delta\epsilon^3\theta$	$+5\alpha\beta\gamma'\delta\epsilon\theta'$	$+15\alpha\beta''\gamma\delta\epsilon\theta'$	$-\alpha\beta\gamma\delta\epsilon\theta\phi^2$
$+6\alpha'\beta\gamma\delta\epsilon\theta$	$-\alpha\beta\gamma\delta\epsilon\theta^3$	$+10\alpha\beta\gamma'\delta\epsilon'\theta$	$+7\alpha'\beta\gamma\delta\epsilon\theta'$	

$$\xi_8 = \alpha\beta\gamma\delta\epsilon\theta\phi\frac{S^8}{8!}+(\alpha\beta\gamma\delta\epsilon\theta\phi'+2\alpha\beta\gamma\delta\epsilon\theta'\phi+3\alpha\beta\gamma\delta\epsilon'\theta\phi+4\alpha\beta\gamma\delta'\epsilon\theta\phi+5\alpha\beta\gamma'\delta\epsilon\theta\phi$$

$$+6\alpha\beta'\gamma\delta\epsilon\theta\phi+7\alpha'\beta\gamma\delta\epsilon\theta\phi)\frac{S^9}{9!}+$$

$$\xi_9 = \alpha\beta\gamma\delta\epsilon\theta\phi\psi.\frac{S^9}{9!}+$$

ERRATA.

Page (i), column 3, line 17, for $+105\alpha'$ read $+105\alpha'^4$.
Page (i), column 4, line 38, for $-\alpha\beta\gamma^4$ read $-\alpha\beta^2\gamma^4$.
Page (ii), column 5, line 15, for $-18\alpha^2\alpha'\beta''$ read $-18\alpha^2\alpha'\beta'''$.
Page (ii), column 4, line 14, for $-7\alpha'\beta^2\gamma'^4$ read $-7\alpha'\beta^2\gamma^4$.
Page (iii), column 2, line 4, for $+7\alpha^{vi}\beta$ read $+7\alpha^{vi}\beta$.
Page (iii), column 2, line 18, for $-4\alpha\beta'\gamma$ read $-4\alpha\beta'\gamma^3$.
Page (iii), column 5, line 14, for $+35\alpha'\beta\gamma\gamma'\delta$ read $+35\alpha'\beta\gamma\gamma'\delta^2$.
Page (iii), column 5, iine 34, for $+22\alpha^3\beta^2\beta\gamma$ read $+22\alpha^3\beta^2\beta'\gamma$.
Page (v), column 1, line 28, for $+\alpha\beta^3\gamma\delta\epsilon$ read $-\alpha\beta^3\gamma\delta\epsilon$.

S. M.